Time-Consistency: from Optimization to Risk Measures

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Abstract

Stochastic optimal control is concerned with sequential decision-making under uncertainty. The theory of dynamic risk measures gives values to stochastic processes (costs) as time goes on and information accumulates. Both theories coin, under the same vocable of *time-consistency* (or *dynamic-consistency*), two different notions: the latter is consistency between successive evaluations of a stochastic processes by a dynamic risk measure as information accumulates (a form of monotonicity); the former is consistency between solutions to intertemporal stochastic optimization problems as information accumulates. Interestingly, time-consistency in stochastic optimal control and time-consistency for dynamic risk measures meet in their use of dynamic programming, or nested, equations. We provide a theoretical framework that offers i) basic ingredients to jointly define dynamic risk measures and corresponding intertemporal stochastic optimization problems ii) common sets of assumptions that lead to time-consistency for both. Our theoretical framework highlights the role of time and risk preferences, materialized in one-step aggregators, in time-consistency. Depending on how you move from one-step time and risk preferences to intertemporal time and risk preferences, and depending on their compatibility (commutation), you will or will not observe time-consistency. We also shed light on the relevance of information structure by giving an explicit role to a state control dynamical system, with a state that parameterizes risk measures and is the input to optimal policies.

Keywords: Dynamic programming, Time consistency, Dynamic risk measures

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Introduction

Stochastic optimal control is concerned with sequential decision-making under uncertainty. The theory of dynamic risk measures gives values to stochastic processes (costs) as time goes on and information accumulates. Both theories coin, under the same vocable of *time-consistency* (or *dynamic-consistency*), two different notions. We discuss one after the other.

In stochastic optimal control, we consider a dynamical process that can be influenced by exogenous noises as well as decisions made at every time step.

The decision maker wants to optimize a criterion (for instance, minimize a net present value) over a given time horizon. As time goes on and the system evolves, observations are made. Naturally, it is generally more profitable for the decision maker to adapt his decisions to the observations on the system. He is hence looking for policies (strategies, decision rules) rather than simple decisions: a policy is a function that maps every possible history of the observations to corresponding decisions.

The notion of “consistent course of action” (see [1]) is well-known in the field of economics, with the seminal work of [2]: an individual having planned his consumption trajectory is consistent if, reevaluating his plans later on, he does not deviate from the originally chosen plan. This idea of consistency as “sticking to one’s plan” may be extended to the uncertain case where plans are replaced by decision rules (“Do thus-and-thus if you find yourself in this portion of state space with this amount of time left”, Richard Bellman cited in [3]): [4] addresses “consistency” and “coherent dynamic choice”, [5] refers to “temporal consistency”.

In this context, we loosely state the property of time-consistency in stochastic optimal control as follows [6]. The decision maker formulates an optimization problem at time t_0 that yields a sequence of optimal decision rules for t_0 and for the following increasing time steps $t_1, \dots, t_N = T$. Then, at the next time step t_1, he formulates a new problem starting at t_1 that yields a new sequence of optimal decision rules from time steps t_1 to T. Suppose the process continues until time T is reached. The sequence of optimization problems is said to be dynamically consistent if the optimal strategies obtained when solving the original problem at time t_0 remain optimal for all subsequent problems. In other words, dynamic consistency means that strategies obtained by solving the problem at the very first stage do not have to be questioned later on.

Now, we turn to dynamic risk measures. At time t_0, you value, by means of a risk measure $\rho_{t_0,T}$, a stochastic process $\big\{\boldsymbol{A}_t\big\}_{t=t_0}^{t_N}$, that represents a stream of costs indexed by the increasing time steps $t_0, t_1, \dots, t_N = T$. Then, at the next time step t_1, you value the tail $\big\{\boldsymbol{A}_t\big\}_{t=t_1}^{t_N}$ of the stochastic process knowing the information obtained and materialized by a σ-field $\mathfrak{F}_{t_1}$. For this, you use a conditional risk measure $\rho_{t_1,T}$ with values in $\mathfrak{F}_{t_1}$-measurable random variables. Suppose the process continues until time T is reached. The sequence $\big\{\rho_{t,T}\big\}_{t=t_0}^{t_N}$ of conditional risk measures is called a dynamic risk measure.

Dynamic or time-consistency has been introduced in the context of risk measures (see [7, 8, 9, 10, 11] for definitions and properties of coherent and consistent dynamic risk measures). We loosely state the property of time-consistency for dynamic risk measures as follows. The dynamic risk measure $\big\{\rho_{t,T}\big\}_{t=t_0}^{t_N}$ is said to be time-consistent when the following property holds. Suppose that two streams of costs, $\big\{\underline{\boldsymbol{A}}_t\big\}_{t=t_0}^{t_N}$ and $\big\{\overline{\boldsymbol{A}}_t\big\}_{t=t_0}^{t_N}$, are such that they coincide from time t_i up to time $t_j > t_i$ and that, from that last time t_j, the tail stream $\big\{\underline{\boldsymbol{A}}_t\big\}_{t=t_j}^{t_N}$ is valued more than $\big\{\overline{\boldsymbol{A}}_t\big\}_{t=t_j}^{t_N}$ $(\rho_{t_j,T}(\big\{\underline{\boldsymbol{A}}_t\big\}_{t=t_j}^{t_N}) \geq \rho_{t_j,T}(\big\{\overline{\boldsymbol{A}}_t\big\}_{t=t_j}^{t_N}))$. Then, the whole stream $\big\{\underline{\boldsymbol{A}}_t\big\}_{t=t_i}^{t_N}$ is valued more than $\big\{\overline{\boldsymbol{A}}_t\big\}_{t=t_i}^{t_N}$ $(\rho_{t_i,T}(\big\{\underline{\boldsymbol{A}}_t\big\}_{t=t_i}^{t_N}) \geq$

$\rho_{t_i,T}(\{\overline{\boldsymbol{A}}_t\}_{t=t_i}^{t_N}))$.

We observe that both notions of time-consistency look quite different: the latter is consistency between successive evaluations of a stochastic processes by a dynamic risk measure as information accumulates (a form of monotonicity); the former is consistency between solutions to intertemporal stochastic optimization problems as information accumulates. We now stress the role of information accumulation in both notions of time-consistency, because of its role in how the two notions can be connected. For dynamic risk measures, the flow of information is materialized by a filtration $\{\mathfrak{F}_t\}_{t=t_1}^{t_N}$. In stochastic optimal control, an amount of information more modest than the past of exogenous noises is often sufficient to make an optimal decision. In the seminal work of [12], the minimal information necessary to make optimal decisions is captured in a *state variable* (see [13] for a more formal definition). Moreover, the famous Bellman or *Dynamic Programming Equation (DPE)* provides a theoretical way to find optimal strategies (see [14] for a broad overview on *Dynamic Programming (DP)*).

Interestingly, time-consistency in stochastic optimal control and time-consistency for dynamic risk measures meet in their use of DPEs. On the one hand, in stochastic optimal control, it is well known that the existence of a DPE with state x for a sequence of optimization problems implies time-consistency when solutions are looked after as feedback policies that are functions of the state x. On the other hand, proving time-consistency for a dynamic risk measure appears rather easy when the corresponding conditional risk measures can be expressed by a *nested* formulation that connects successive time steps. In both contexts, such nested formulations are possible only for proper information structures. In stochastic optimal control, a sequence of optimization problems may be consistent for some information structure while inconsistent for a different one (see [6]). For dynamic risk measures, time-consistency appears to be strongly dependent on the underlying information structure (filtration or scenario tree). Moreover, in both contexts, nested formulations and the existence of a DPE are established under various forms of decomposability of operators that display monotonicity and commutation properties.

Our objective is to provide a theoretical framework that offers i) basic ingredients to jointly define dynamic risk measures and corresponding intertemporal stochastic optimization problems ii) common sets of assumptions that lead to time-consistency for both. Our theoretical framework highlights the role of time and risk preferences, materialized in *one-step aggregators*, in time-consistency. Depending on how you move from one-step time and risk preferences to intertemporal time and risk preferences, and depending on their compatibility (commutation), you will or will not observe time-consistency. We also shed light on the relevance of information structure by giving an explicit role to a dynamical system with state $\boldsymbol{X}$.

In §1, we present examples of intertemporal optimization problems displaying a DPE, and of dynamic risk measures (time-consistent or not, nested or not). In §2, we introduce the basic material to formulate intertemporal optimization problems, in the course of which we define "cousins" of dynamic risk measures, namely *dynamic uncertainty criteria*; we end with definitions of time-consistency, on the one hand, for dynamic risk measures and, in the other hand, for intertemporal stochastic optimization problems. In §3, we introduce the notions of time and uncertainty-aggregators, define their composition, and show four ways to craft a dynamic uncertainty criterion from one-step aggregators; then, we provide general sufficient conditions for the existence of a DPE and for time-consistency, both for dynamic risk measures and for intertemporal stochastic optimization problems; we end with applications. In §4, we extend constructions and results to Markov aggregators.

1. Introductory Examples

The traditional framework for DP consists in minimizing the expectation of the intertemporal sum of costs as in Problem (3). As we see it, the intertemporal sum is an aggregation over time, and the mathematical expectation is an aggregation over uncertainties. We claim that other forms of aggregation lead to a DPE with the same state but, before developing this point in §3, we lay out in §1.1 three settings (more or less familiar) in which a DPE holds. We do the same job for dynamic risk measures in §1.2 with time-consistency.

To alleviate notations, for any sequence $\big\{H_s\big\}_{s=t_1,\dots,t_2}$ of sets, we denote by $\big[H_s\big]_{t_1}^{t_2}$, or by $H_{[t_1:t_2]}$, the Cartesian product

$$H_{[t_1:t_2]} = \big[H_s\big]_{t_1}^{t_2} = \big[H_s\big]_{s=t_1}^{t_2} = H_{t_1} \times \cdots \times H_{t_2} \ , \tag{1a}$$

and a generic element by

$$h_{[t_1:t_2]} = \big\{h_t\big\}_{t_1}^{t_2} = \big\{h_t\big\}_{t=t_1}^{t_2} = (h_{t_1}, \dots, h_{t_2}) \ . \tag{1b}$$

In the same vein, we also use the following notation for any sequence

$$H_{[t_1:t_2]} = \big\{H_s\big\}_{t_1}^{t_2} = \big\{H_s\big\}_{s=t_1}^{t_2} = \big\{H_s\big\}_{s=t_1,\dots,t_2} \ . \tag{1c}$$

In this chapter, we denote by $\bar{\mathbb{R}}$ the set $\mathbb{R} \cup \{+\infty\}$.

1.1. Examples of DPEs in Intertemporal Optimization

Anticipating on material to be presented in §2.1, we consider a dynamical system influenced by exogenous uncertainties and by decisions made at discrete time steps $t = 0$, $t = 1$, …, $t = T - 1$, where T is a positive integer. For any $t \in [\![0, T]\!]$, where $[\![a, b]\!]$ denote the set of integers between a and b, we suppose given a state set $\mathbb{X}_t$, and for any $t \in [\![0, T-1]\!]$ a control set $\mathbb{U}_t$, an uncertainty set $\mathbb{W}_t$ and a mapping f_t that maps $\mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t$ into $\mathbb{X}_{t+1}$. We consider the *control stochastic dynamical system*

$$\forall t \in [\![0, T-1]\!], \qquad X_{t+1} = f_t(X_t, U_t, W_t) \ . \tag{2}$$

We call *policy* a sequence $\pi = (\pi_t)_{t\in[\![0,T-1]\!]}$ of mappings where, for all $t \in [\![0, T-1]\!]$, π_t maps $\mathbb{X}_t$ into $\mathbb{U}_t$. We denote by Π the set of all policies. More generally, for all $t \in [\![0, T]\!]$, we call *(tail) policy* a sequence $\pi = (\pi_s)_{s\in[\![t,T-1]\!]}$ and we denote by Π_t the set of all such policies.

Let $\{\boldsymbol{W}_t\}_{t=0}^{T}$ be a sequence of independent random variables (noises). Let $\{J_t\}_0^{T-1}$ be a sequence of cost functions $J_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \mapsto \mathbb{R}$, and a final cost function $J_T : \mathbb{X}_T \times \mathbb{W}_T \to \mathbb{R}$.

With classic notations, and assuming all proper measurability and integrability conditions, we consider the dynamic optimization problem

$$\min_{\pi\in\Pi} \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} J_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t) + J_T(\boldsymbol{X}_T, \boldsymbol{W}_T)\Big] , \tag{3a}$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t), \qquad \forall t \in [\![0, T-1]\!] , \tag{3b}$$

$$\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t), \qquad \forall t \in [\![0, T-1]\!] . \tag{3c}$$

It is well-known that a DPE with state $\boldsymbol{X}$ can be associated with this problem. The main ingredients for establishing the DPE are the following: the intertemporal criterion is time-separable and additive, the expectation is a composition of expectations over the marginals law (because the random variables $\{\boldsymbol{W}_t\}_{t=0}^{T}$ are independent), and the sum and the expectation operators are commuting. Our main concern is to extend these properties to other "aggregators" than the intertemporal sum $\sum_{t=0}^{T-1}$ and the mathematical expectation $\mathbb{E}$, and to obtain DPEs with state $\boldsymbol{X}$, thus retrieving time-consistency.

In this example, we aggregate the streams of cost first with respect to time (through the sum over the stages), and then with respect to uncertainties (through the expectation). This formulation is called *TU* for "time then uncertainty". All the examples of this §1.1 follow this template.

We do not present proofs of the DPEs exposed here as they fit into the framework developed later in §3.

1.1.1. Expected and Worst Case with Additive Costs

We present together two settings in which a DPE holds true. They share the same time-aggregator — time-separable and additive — but with distinct uncertainty-aggregators, namely the mathematical expectation operator and the so-called "fear" operator.

Expectation Operator. Consider, for any $t \in [\![0, T]\!]$, a probability $\mathbb{P}_t$ on the uncertainty space $\mathbb{W}_t$ (equipped with a proper σ-algebra), and the product probability $\mathbb{P} = \mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_T$. In other formulations of stochastic optimization problems, the probabilities $\mathbb{P}_t$ are the image distributions of independent random variables with value in $\mathbb{W}_t$. However, we prefer to ground the problems with probabilities on the uncertainty spaces rather than with random variables, as this approach will more easily easily extend to other contexts without stochasticity.

The so-called *value function* V_t, whose argument is the state x, is the optimal cost-to-go defined by

$$
\begin{aligned}
V_t(x) = \min_{\pi \in \Pi_t} \quad & \mathbb{E}\Bigg[\sum_{s=t}^{T-1} J_s(\boldsymbol{X}_s, \boldsymbol{U}_s, \boldsymbol{W}_s) + J_T(\boldsymbol{X}_T, \boldsymbol{W}_T) \Bigg] , && \text{(4a)} \\
\text{s.t.} \quad & \boldsymbol{X}_t = x \, , && \text{(4b)} \\
& \boldsymbol{X}_{s+1} = f_t(\boldsymbol{X}_s, \boldsymbol{U}_s, \boldsymbol{W}_s), \qquad \forall s \in [\![t, T-1]\!] \, , && \text{(4c)} \\
& \boldsymbol{U}_s = \pi_s(\boldsymbol{X}_s) \, . && \text{(4d)}
\end{aligned}
$$

The DPE associated with problem (3) is

$$
\begin{cases}
V_T(x) & = \mathbb{E}_{\mathbb{P}_T}\Big[J_T(x, \boldsymbol{W}_T) \Big] \, , \\
V_t(x) & = \min_{u \in \mathbb{U}_t} \mathbb{E}_{\mathbb{P}_t}\Big[J_t(x, u, \boldsymbol{W}_t) + V_{t+1} \circ f_t(x, u, \boldsymbol{W}_t) \Big] \, ,
\end{cases}
\tag{5}
$$

for all state $x \in \mathbb{X}_t$ and all time $t \in [\![0, T-1]\!]$.

It is well-known that, if there exists a policy $\pi^\sharp$ (with proper measurability assumptions that we do not discuss here [see [15]]) such that, for each $t \in [\![0, T-1]\!]$, and each $x \in \mathbb{X}_t$, we have

$$
\pi_t^\sharp(x) \in \operatorname*{arg\,min}_{u \in \mathbb{U}_t} \mathbb{E}\Big[J_t(x, u, \boldsymbol{W}_t) + V_{t+1} \circ f_t(x, u, \boldsymbol{W}_t) \Big] \, , \tag{6}
$$

then $\pi^\sharp$ is an optimal policy for Problem (3).

Time-consistency of the sequence of Problems (4), when t runs from 0 to T, is ensured by this very DPE, when solutions are looked after as policies over the state x. We insist that the property of time-consistency may or may not hold depending on the nature of available information at each time step. Here, our assumption is that the state x_t is available for decision-making at each time t.[1]

Remark 1. *To go on with information issues, we can notice that the so-called "non-anticipativity constraints", typical of stochastic optimization, are contained in our definition of policies. Indeed, we considered policies are function of the state, which a summary of the* past, *hence cannot anticipate the future. Why can we take the state as a proper summary? If, in Problem* (3), *we had considered policies as functions of past uncertainties (non-anticipativity) and had assumed that the uncertainties are independent, it is well-known that we could have restricted our search to optimal Markovian policies, that is, only functions of the state. This is why, we consider policies only as functions of the state.*

[1] In the literature on risk measures, information is rather described by filtrations than by variables.

Fear Operator. In [16], Pierre Bernhard coined *fear operator* the worst-case operator, widely considered in the field of robust optimization (see [17] and [18]).

We consider the optimization problem

$$\min_{\pi\in\Pi} \quad \sup_{w\in\mathbb{W}_{[0:T]}} \Big[\sum_{t=0}^{T-1} J_t(x_t,u_t,w_t) + J_T(x_T,w_T)\Big], \tag{7a}$$

$$\text{s.t.} \quad x_{t+1} = f_t(x_t,u_t,w_t), \tag{7b}$$

$$u_t = \pi_t(x_t). \tag{7c}$$

Contrarily to previous examples we do not use bold letters for state x, control u and uncertainty w as these variables are not random variables. In [19, Section 1.6], it is shown that the value function

$$V_t(x) = \min_{\pi\in\Pi_t} \quad \sup_{w\in\mathbb{W}_{[t:T]}} \left[\sum_{s=t}^{T-1} J_s(x_s,u_s,w_s) + J_T(x_T,w_T)\right], \tag{8a}$$

$$\text{s.t.} \quad x_t = x\ , \tag{8b}$$

$$x_{s+1} = f_s(x_s,u_s,w_s)\ , \tag{8c}$$

$$u_s = \pi_s(x_s)\ . \tag{8d}$$

satisfies the DPE

$$\begin{cases} V_T(x) = \sup\limits_{w_T\in\mathbb{W}_T} J_T(x,w_T)\ , \\ V_t(x) = \min\limits_{u\in\mathbb{U}_t} \sup\limits_{w_t\in\mathbb{W}_t} \Big[J_t(x,u,w_t) + V_{t+1}\circ f_t(x,u,w_t)\Big]\ , \end{cases} \tag{9}$$

for all state $x \in \mathbb{X}_t$ and all time $t \in [\![0, T-1]\!]$.

1.1.2. Expectation with Multiplicative Costs

An expected multiplicative cost appears in a financial context if we consider a final payoff $K(\boldsymbol{X}_{T+1})$ depending on the final state of our system, but discounted at rate $r_t(\boldsymbol{X}_t)$. In this case, the problem of maximizing the discounted expected product reads

$$\max_{\pi\in\Pi} \quad \mathbb{E}\Big[\prod_{t=1}^{T-1}\Big(\frac{1}{1+r_t(\boldsymbol{X}_t)}\Big)K(\boldsymbol{X}_T)\Big]\ .$$

We present another interesting setting where multiplicative cost appears. In control problems, we thrive to find controls such that the state x_t satisfies constraints of the type $x_t \in \mathcal{X}_t \subset \mathbb{X}_t$ for all $t \in [\![0,T]\!]$. In a deterministic setting, the problem has either no solution (there is no policy such that, for all $t \in [\![0,T]\!]$, $x_t \in \mathcal{X}_t$) or has a solution depending on the starting point x_0. However, in a stochastic setting, satisfying the constraint $x_t \in \mathcal{X}_t$, for all time $t \in [\![0,T]\!]$ and $\mathbb{P}-$almost surely, can lead to problems without solution. For example, if we add to a controled dynamic a nondegenerate Gaussian random variable, then the resulting state can be anywhere in the state space, and thus a constraint $\boldsymbol{X}_t \in \mathcal{X}_t \subset \mathbb{X}_t$ where $\mathcal{X}_t$ is, say, a bounded set, cannot be satisfied almost surely.

For such a control problem, we propose alternatively to maximize the probability of satisfying the constraint (see [20], where this is approach is called *stochastic viability*):

$$\max_{\pi \in \Pi} \quad \mathbb{P}\Big(\big\{\forall t \in [\![0, T]\!] \;,\;\; \boldsymbol{X}_t \in \mathcal{X}_t\big\}\Big), \tag{10a}$$

$$\text{s.t} \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \;, \tag{10b}$$

$$\boldsymbol{U}_t = \pi(\boldsymbol{X}_t) \;. \tag{10c}$$

This problem can be written

$$\max_{\pi \in \Pi} \quad \mathbb{E}\Big[\prod_{t=0}^{T} \mathbb{1}_{\{\boldsymbol{X}_t \in \mathcal{X}_t\}}\Big], \tag{11a}$$

$$\text{s.t} \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \;, \tag{11b}$$

$$\boldsymbol{U}_t = \pi(\boldsymbol{X}_t) \;. \tag{11c}$$

It is shown in [21] that, assuming that noises are independent (i.e the probability $\mathbb{P}$ can be written as a product $\mathbb{P} = \mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_T$), the associated DPE is

$$\left\{\begin{array}{ll} V_T(x) & = \mathbb{E}\Big[\mathbb{1}_{\{x \in \mathcal{X}_T\}}\Big] \;, \\ V_t(x) & = \max_{u \in \mathbb{U}_t} \quad \mathbb{E}\Big[\mathbb{1}_{\{x \in \mathcal{X}_t\}} \cdot V_{t+1} \circ f_t(x, u, \boldsymbol{W}_t)\Big] \;, \end{array}\right. \tag{12}$$

for all state $x \in \mathbb{X}_t$ and all time $t \in [\![0, T-1]\!]$.

If there exists a measurable policy $\pi^\sharp$ such that, for all $t \in [\![0, T-1]\!]$ and all $x \in \mathbb{X}_t$,

$$\pi_t^\sharp(x) \in \operatorname*{arg\,max}_{u \in \mathbb{U}_t} \quad \mathbb{E}\Big[\mathbb{1}_{\{x \in \mathcal{X}_t\}} \cdot V_{t+1} \circ f_t(x, u, \boldsymbol{W}_t)\Big] \;, \tag{13}$$

then $\pi^\sharp$ is optimal for Problem (10).

1.2. Examples of Dynamic Risk Measures

Consider a probability space $\big(\Omega, \mathfrak{F}, \mathbb{P}\big)$, and a filtration $\mathfrak{F} = \{\mathfrak{F}_t\}_0^T$. The expression $\{\boldsymbol{A}_s\}_0^T$ denotes an arbitrary, $\mathfrak{F}$-adapted, real-valued, stochastic process.

Anticipating on recalls in §2.2.2, we call *conditional risk measure* a function $\rho_{t,T}$ that maps tail sequences $\{\boldsymbol{A}_s\}_t^T$, where each $\boldsymbol{A}_s$ is $\mathfrak{F}_s$ measurable, into the set of $\mathfrak{F}_t$ measurable random variables. A *dynamic risk measure* is a sequence $\{\rho_{t,T}\}_0^T$ of conditional risk measures.

A dynamic risk measure $\{\rho_{t,T}\}_{t=0}^T$, is said to be *time-consistent* if, for any couples of times $0 \leq \underline{t} < \overline{t} \leq T$, the following property holds true. If two adapted stochastic processes $\{\underline{\boldsymbol{A}}_s\}_0^T$ and $\{\overline{\boldsymbol{A}}_s\}_0^T$ satisfy

$$\underline{\boldsymbol{A}}_s = \overline{\boldsymbol{A}}_s \;, \qquad \forall s \in [\![\underline{t}, \overline{t} - 1]\!] \;, \tag{14a}$$

$$\rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \leq \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \;, \tag{14b}$$

then we have:

$$\rho_{\underline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\underline{t}}^{T}\big) \leq \rho_{\underline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\underline{t}}^{T}\big) \ . \tag{14c}$$

We now lay out examples of dynamic risk measure.

1.2.1. Expectation and Sum

Unconditional Expectation. The first classical example, related to the optimization Problem (3), consists in the dynamic risk measure $\{\rho_{t,T}\}_{t=0}^{T}$ given by

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathbb{E}\Big[\sum_{s=t}^{T} \boldsymbol{A}_s\Big] \ , \ \ \forall t \in [\![0,T]\!] \ . \tag{15}$$

We write (15) under three forms — denoted by TU, UT, NTU, and discussed later in §3.1:

$$\begin{aligned}
\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) &= \mathbb{E}\Big[\sum_{s=t}^{T} \boldsymbol{A}_s\Big] && (TU)\\
&= \sum_{s=t}^{T} \mathbb{E}\big[\boldsymbol{A}_s\big] && (UT)\\
&= \mathbb{E}\bigg[\boldsymbol{A}_t + \mathbb{E}\Big[\boldsymbol{A}_{t+1} + \cdots + \mathbb{E}\Big[\boldsymbol{A}_{T-1} + \mathbb{E}\big[\boldsymbol{A}_T\big]\Big]\cdots\Big]\bigg] && (NTU)
\end{aligned}$$

To illustrate the notion, we show that the dynamic risk measure $\{\rho_{t,T}\}_{t=0}^{T}$ is time-consistent. Indeed, if two adapted stochastic processes $\underline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ satisfy (14a) and (14b), with $\underline{t} = t < \overline{t} \leq T$, we conclude that

$$\begin{aligned}
\rho_{t,T}\big(\{\underline{\boldsymbol{A}}_s\}_t^T\big) =& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1} \underline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big)\Big]\\
\leq& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1} \overline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big)\Big] = \rho_{t,T}\big(\{\overline{\boldsymbol{A}}_s\}_t^T\big) \ .
\end{aligned}$$

Conditional Expectation. Now, we consider a "conditional variation" of (15) by defining

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathbb{E}\Big[\sum_{s=t}^{T} \boldsymbol{A}_s \ \Big| \ \mathfrak{F}_t\Big] \ . \tag{16}$$

We write[2] the induced dynamic risk measure $\{\rho_{t,T}\}_{t=0}^{T}$ under four forms — denoted by TU, UT, NTU, NUT, and discussed later in §3.1:

$$\begin{aligned}
\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) &= \mathbb{E}^{\mathfrak{F}_t}\Big[\sum_{s=t}^{T}\boldsymbol{A}_s\Big] && (TU)\\
&= \sum_{s=t}^{T}\mathbb{E}^{\mathfrak{F}_t}\big[\boldsymbol{A}_s\big] && (UT)\\
&= \mathbb{E}^{\mathfrak{F}_t}\Bigg[\boldsymbol{A}_t + \mathbb{E}^{\mathfrak{F}_{t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots + \mathbb{E}^{\mathfrak{F}_{T-1}}\Big[\boldsymbol{A}_{T-1} + \mathbb{E}^{\mathfrak{F}_T}\big[\boldsymbol{A}_T\big]\Big]\cdots\Big]\Bigg] && (NTU)\\
&= \boldsymbol{A}_t + \mathbb{E}^{\mathfrak{F}_{t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots + \mathbb{E}^{\mathfrak{F}_{T-2}}\big[\boldsymbol{A}_{T-1} + \mathbb{E}^{\mathfrak{F}_{T-1}}\big[\boldsymbol{A}_T\big]\big]\cdots\Big] && (NUT)
\end{aligned}$$

The dynamic risk measure $\{\rho_{t,T}\}_{t=0}^{T}$ is time-consistent: indeed, if two adapted stochastic processes $\underline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ satisfy (14a) and (14b), with $\underline{t} = t < \overline{t} \leq T$, we conclude that

$$\begin{aligned}
\rho_{t,T}\big(\{\underline{\boldsymbol{A}}_s\}_t^T\big) = \;& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1}\underline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \;\Big|\; \mathfrak{F}_t\Big]\\
\leq \;& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1}\overline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \;\Big|\; \mathfrak{F}_t\Big] = \rho_{t,T}\big(\{\overline{\boldsymbol{A}}_s\}_t^T\big)\;.
\end{aligned}$$

1.2.2. AV@R and Sum

In the following examples, it is no longer possible to display three or four equivalent expressions for the same conditional risk measure. This is why, we present different dynamic risk measures.

Unconditional AV@R. For $0 < \alpha < 1$, we define the *Average-Value-at-Risk* of level α of a random variable $\boldsymbol{X}$ by

$$\mathrm{AV@R}_{\alpha}\big[\boldsymbol{X}\big] = \inf_{r\in\mathbb{R}}\Big\{r + \frac{\mathbb{E}\big[\boldsymbol{X} - r\big]^{+}}{\alpha}\Big\}\;. \tag{17}$$

Let $\big\{\alpha_t\big\}_{t=0}^{T}$ and $\big\{\alpha_{t,s}\big\}_{s,t=0}^{T}$ be two families in $(0,1)$. We lay out three different dynamic risk measures, given by the following conditional risk measures:

[2] Here, for notational clarity, we denote by $\mathbb{E}^{\mathfrak{F}_t}\big[\,\cdot\,\big]$ the conditional expectation $\mathbb{E}\big[\,\cdot \;\big|\; \mathfrak{F}_t\big]$.

$$\rho_{t,T}\big[\{\boldsymbol{A}_s\}_t^T\big] = \mathrm{AV@R}_{\alpha_t}\Big[\sum_{s=t}^{T}\boldsymbol{A}_s\Big] \,, \tag{TU}$$

$$\rho_{t,T}\big[\{\boldsymbol{A}_s\}_t^T\big] = \sum_{s=t}^{T}\mathrm{AV@R}_{\alpha_{t,s}}\big[\boldsymbol{A}_s\big] \,, \tag{UT}$$

$$\rho^{NTU}_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathrm{AV@R}_{\alpha_{t,t}}\Big[\boldsymbol{A}_t + \mathrm{AV@R}_{\alpha_{t,t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots \\ \mathrm{AV@R}_{\alpha_{t,T}}\big[\boldsymbol{A}_T\big]\cdots\Big]\Big] \,. \tag{UT}$$

The dynamic risk measure $\{\rho^{TU}_{t,T}\}_{t=0}^T$ is not time-consistent, whereas the dynamic risk measure $\{\rho^{UT}_{t,T}\}_{t=0}^T$ and the dynamic risk measure $\{\rho^{NTU}_{t,T}\}_{t=0}^T$ are time consistent, as soon as the levels $\alpha_{t,s}$ do not depend on t.

Conditional AV@R. For $0<\alpha<1$, and a subfield $\mathfrak{G}\subset\mathfrak{F}$ we define the *conditional Average-Value-at-Risk* of level α of a random variable $\boldsymbol{X}$ knowing $\mathfrak{G}$ by

$$\mathrm{AV@R}^{\mathfrak{G}}_{\alpha}\big[\boldsymbol{X}\big] = \inf_{r\ \mathfrak{G}\text{-measurable}}\Big\{r + \frac{\mathbb{E}\big[\boldsymbol{X}-r \mid \mathfrak{G}\big]^+}{\alpha}\Big\} \,. \tag{19}$$

Let $\{\alpha_t\}_{t=0}^T$ and $\{\alpha_{t,s}\}_{s,t=0}^T$ be two families in $(0,1)$. We lay out four different dynamic risk measures, given by the following conditional risk measures:

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathrm{AV@R}^{\mathfrak{F}_t}_{\alpha_t}\Big[\sum_{s=t}^{T}\boldsymbol{A}_s\Big] \,, \tag{TU}$$

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \sum_{s=t}^{T}\mathrm{AV@R}^{\mathfrak{F}_t}_{\alpha_{t,s}}\big[\boldsymbol{A}_s\big] \,, \tag{UT}$$

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \sum_{s=t}^{T}\mathrm{AV@R}^{\mathfrak{F}_t}_{\alpha_{t,t}}\Big[\mathrm{AV@R}^{\mathfrak{F}_{t+1}}_{\alpha_{t,t+1}}\Big[\cdots\mathrm{AV@R}^{\mathfrak{F}_s}_{\alpha_{t,s}}\big[\boldsymbol{A}_s\big]\Big]\Big] \,, \tag{UT}$$

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathrm{AV@R}^{\mathfrak{F}_t}_{\alpha_{t,t}}\Big[\boldsymbol{A}_t + \\ \mathrm{AV@R}^{\mathfrak{F}_{t+1}}_{\alpha_{t,t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots\mathrm{AV@R}^{\mathfrak{F}_T}_{\alpha_{t,T}}\big[\boldsymbol{A}_T\big]\cdots\Big]\Big] \,. \tag{NTU}$$

Examples of this type are found in papers like [22, 23, 24, 25].

Markovian AV@R. Let a policy $\pi\in\Pi$, a time $t\in[\![0,T]\!]$ and a state $x_t\in\mathbb{X}_t$ be fixed. With this and the control stochastic dynamical system (2), we define the

Markov chain $\{\boldsymbol{X}_s^{x_t}\}_{s=t}^T$ produced by (3b)–(3c) starting from $\boldsymbol{X}_t = x_t$. We also define, for each $s \in [\![t, T]\!]$, the σ-algebra $\mathcal{X}_s^{x_t} = \sigma(\boldsymbol{X}_s^{x_t})$. With this, we define a conditional risk measure by

$$\begin{aligned}\rho_{t,T}^{x_t}\big(\{\boldsymbol{A}_s\}_t^T\big) =& \mathrm{AV@R}_{\alpha_{t,t}}^{\mathcal{X}_t^{x_t}}\Bigg[\boldsymbol{A}_t + \\ & \mathrm{AV@R}_{\alpha_{t,t+1}}^{\mathcal{X}_{t+1}^{x_t}}\Big[\boldsymbol{A}_{t+1} + \cdots \mathrm{AV@R}_{\alpha_{t,T}}^{\mathcal{X}_T^{x_t}}\big[\boldsymbol{A}_T\big]\cdots\Big]\Bigg] \, .\end{aligned} \tag{21}$$

Repeating the process, we obtain a family $\Big\{\big\{\varrho_{t,T}^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^T$, such that $\big\{\varrho_{t,T}^{x_t}\big\}_{t=0}^T$ is a dynamic uncertainty criterion, for all sequence $\big\{x_t\big\}_{t=0}^T$ of states, where $x_t \in \mathbb{X}_t$, for all $t \in [\![0, T]\!]$.

2. Time-Consistency: Problem Statement

In §2.1, we lay out the basic material to formulate intertemporal optimization problems. In §2.2, we define "cousins" of dynamic risk measures, namely *dynamic uncertainty criteria*. In §2.3, we provide definitions of time-consistency, on the one hand, for dynamic risk measures and, in the other hand, for intertemporal stochastic optimization problems.

2.1. Ingredients for Intertemporal Optimization Problems

In §2.1.1, we recall the formalism of Control Theory, with dynamical system, state, control and costs. Mimicking the definition of adapted processes in Probability Theory, we introduce adapted uncertainty processes. In §2.1.2, we show how to produce an adapted uncertainty process of costs.

2.1.1. Dynamical System, State, Control and Costs

We define a *control T-stage dynamical system*, with $T \geq 2$, as follows. We consider

- a sequence $\big\{\mathbb{X}_t\big\}_0^T$ of sets of *states*;
- a sequence $\big\{\mathbb{U}_t\big\}_0^{T-1}$ of sets of *controls*;
- a sequence $\big\{\mathbb{W}_t\big\}_0^T$ of sets of *uncertainties*, and we define

$$\mathbb{W}_{[0:T]} = \big[\mathbb{W}_s\big]_0^T \, , \quad \text{the set of } \textit{scenarios}, \tag{22a}$$

$$\mathbb{W}_{[0:t]} = \big[\mathbb{W}_s\big]_0^t \, , \quad \text{the set of } \textit{head scenarios}, \ \forall t \in [\![0, T]\!] \, , \tag{22b}$$

$$\mathbb{W}_{[s:t]} = \big[\mathbb{W}_s\big]_t^T \, , \quad \text{the set of } \textit{tail scenarios}, \ \forall t \in [\![0, T]\!] \, ; \tag{22c}$$

- a sequence $\big\{f_t\big\}_0^{T-1}$ of *functions*, where $f_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \to \mathbb{X}_{t+1}$, to play the role of *dynamics*;

- a sequence $\{U_t\}_0^{T-1}$ of T *multifunctions* $U_t : \mathbb{X}_t \rightrightarrows \mathbb{U}_t$, to play the role of *constraints*;
- a sequence $\{J_t\}_0^{T-1}$ of *cost functions* $J_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \mapsto \bar{\mathbb{R}}$, and a final cost function $J_T : \mathbb{X}_T \times \mathbb{W}_T \to \bar{\mathbb{R}}$.[3]

Mimicking the definition of adapted processes in Probability Theory, we introduce the following definition of *adapted uncertainty processes*, where the increasing sequence of head scenarios sets in (22b) corresponds to a filtration.

Definition 1. We say that a sequence $A_{[0:T]} = \{A_s\}_0^T$ is an *adapted uncertainty process* if $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}\big)$ (that is, $A_s : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$), for all $s \in [\![0, T]\!]$. In other words, $\big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})\big]_{s=0}^T$ is the set of adapted uncertainty processes.

A *policy* $\pi = (\pi_t)_{t \in [\![0,T-1]\!]}$ is a sequence of functions $\pi_t : \mathbb{X}_t \to \mathbb{U}_t$, and we denote by Π the set of all policies. More generally, for all $t \in [\![0, T]\!]$, we call *(tail) policy* a sequence $\pi = (\pi_s)_{s \in [\![t,T-1]\!]}$ and we denote by Π_t the set of all such policies.

We restrict our search of optimal solutions to so-called *admissible* policies belonging to a subset $\Pi^{\mathrm{ad}} \subset \Pi$. An admissible policy $\pi \in \Pi^{\mathrm{ad}}$ always satisfies:

$$\forall t \in [\![0, T-1]\!] \ , \ \ \forall x \in \mathbb{X}_t, \qquad \pi_t(x) \in U_t(x) \ .$$

We can express in Π^{ad} other types of constraints, such as measurability or integrability ones when in a stochastic setting. Naturally, we set $\Pi_t^{ad} = \Pi_t \cap \Pi^{ad}$.

Definition 2. For any time $t \in [\![0, T]\!]$, state $x \in \mathbb{X}_t$ and policy $\pi \in \Pi$, the *flow* $\{X_{t,s}^{x,\pi}\}_{s=t}^T$ is defined by the forward induction:

$$\forall w \in \mathbb{W}_{[0:T]} \ , \ \begin{cases} X_{t,t}^{x,\pi}(w) & = x \ , \\ X_{t,s+1}^{x,\pi}(w) & = f_s\big(X_{t,s}^{x,\pi}(w), \pi_s(X_{t,s}^{x,\pi}(w)), w_s\big) \ , \ \ \forall s \in [\![t, T]\!] \ . \end{cases} \tag{23}$$

The expression $X_{t,s}^{x,\pi}(w)$ is the state $x_s \in \mathbb{X}_s$ reached at time $s \in [\![0, T]\!]$, when starting at time $t \in [\![0, s]\!]$ from state $x \in \mathbb{X}_t$ and following the dynamics (2) with the policy $\pi \in \Pi$ along the scenario $w \in \mathbb{W}_{[0:T]}$.

Remark 2. *For $0 \leq t \leq s \leq T$, the flow $X_{t,s}^{x,\pi}$ is a function that maps the set $\mathbb{W}_{[0:T]}$ of scenarios into the state space $\mathbb{X}_s$:*

$$X_{t,s}^{x,\pi} : \mathbb{W}_{[0:T]} \to \mathbb{X}_s \ . \tag{24}$$

By (23),

[3] For notational consistency with the J_t for $t = [\![0, T-1]\!]$, we will often write $J_T(x, u, w)$ to mean $J_T(x, w)$.

- *when $t > 0$, the expression $X^{x,\pi}_{t,s}(w)$ depends only on the inner part $w_{[t:s-1]}$ of the scenario $w = w_{[0:T]}$, hence depends neither on the head $w_{[0:t-1]}$, nor on the tail $w_{[s:T]}$,*
- *when $t = 0$, the expression $X^{x,\pi}_{0,s}(w)$ in (23) depends only on the head $w_{[0:s-1]}$ of the scenario $w = w_{[0:T]}$, hence does not depend on the tail $w_{[s:T]}$.*

This is why we often consider that the flow $X^{x,\pi}_{t,s}$ is a function that maps the set $\mathbb{W}_{[t:s-1]}$ of scenarios into the state space $\mathbb{X}_s$:

$$\forall s \in [\![1,T]\!] \ , \ \ \forall t \in [\![0,s-1]\!], \qquad X^{x,\pi}_{t,s} : \mathbb{W}_{[t:s-1]} \to \mathbb{X}_s \ . \tag{25}$$

A *state trajectory* is a realization of the flow $\left\{X^{x,\pi}_{0,s}(w)\right\}_{s=0}^{T}$ for a given scenario $w \in \mathbb{W}_{[0:T]}$. The *flow property*

$$\forall t,s,s' \ , \ \ t < s' < s \ , \ \ \forall x \in \mathbb{X}_t \ , \ \ X^{x,\pi}_{t,s} \equiv X^{X^{x,\pi}_{t,s'},\pi}_{s',s} \tag{26}$$

expresses the fact that we can stop anywhere along a state trajectory and start again.

2.1.2. Producing Streams of Costs

Definition 3. For a given policy $\pi \in \Pi$, and for all times $t \in [\![0,T]\!]$ and $s \in [\![t,T]\!]$, we define the uncertain costs evaluated along the state trajectories by:

$$J^{x,\pi}_{t,s} : \quad w \in \mathbb{W}_{[0:T]} \longmapsto J_s\Big(X^{x,\pi}_{t,s}(w), \pi\big(X^{x,\pi}_{t,s}(w)\big), w_s\Big) \ . \tag{27}$$

Remark 3. *By Remark 2,*

- *when $t > 0$, the expression $J^{x,\pi}_{t,s}(w)$ in (27) depends only on the inner part $w_{[t:s]}$ of the scenario $w = w_{[0:T]}$, hence depends neither on the head $w_{[0:t-1]}$, nor on the tail $w_{[s+1:T]}$,*
- *when $t = 0$, the expression $J^{x,\pi}_{0,s}(w)$ in (27) depends only on the head $w_{[0:s]}$ of the scenario $w = w_{[0:T]}$, hence does not depend on the tail $w_{[s+1:T]}$.*

This is why we often consider that $J^{x,\pi}_{t,s}$ is a function that maps the set $\mathbb{W}_{[t:s]}$ of scenarios into $\bar{\mathbb{R}}$:

$$\forall s \in [\![0,T]\!] \ , \ \ \forall t \in [\![0,s]\!], \qquad J^{x,\pi}_{t,s} : \quad \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}} \ . \tag{28}$$

As a consequence, the stream $\left\{J^{x,\pi}_{0,s}\right\}_{s=0}^{T}$ of costs is an adapted uncertainty process.

By (27) and (23), we have, for all $t \in [\![0,T]\!]$ and $s \in [\![t+1,T]\!]$,

$$\forall w_{[t:T]} \in \mathbb{W}_{[t:T]} \ , \ \begin{cases} J^{x,\pi}_{t,t}(w_t) &= J_t\big(x, \pi_t(x), w_t\big) \ , \\ J^{x,\pi}_{t,s}(w_t, \{w_r\}_{t+1}^{T}) &= J^{f_t(x,\pi_t(x),w_t),\pi}_{t+1,s}(\{w_r\}_{t+1}^{T}) \ . \end{cases} \tag{29}$$

2.2. Dynamic Uncertainty Criteria and Dynamic Risk Measures

Now, we stand with a stream $\left\{J_{0,s}^{x;\pi}\right\}_{s=0}^{T}$ of costs, which is an adapted uncertainty process by Remark 2. To craft a criterion to optimize, we need to aggregate such a stream into a scalar. For this purpose, we define *dynamic uncertainty criterion* in §2.2.1, and relate them to dynamic risk measures in §2.2.2.

2.2.1. Dynamic Uncertainty Criterion

Inspired by the definitions of risk measures and dynamic risk measures in Mathematical Finance, and motivated by intertemporal optimization, we introduce the following definitions of *dynamic uncertainty criterion*, and *Markov dynamic uncertainty criterion*. Examples have been given in §1.2.

Definition 4. A *dynamic uncertainty criterion* is a sequence $\{\varrho_{t,T}\}_{t=0}^{T}$, such that, for all $t \in [\![0,T]\!]$,

- $\varrho_{t,T}$ is a mapping

$$\varrho_{t,T} : \left[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\right]_{s=t}^{T} \to \mathcal{F}(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}) \; , \tag{30a}$$

- the restriction of $\varrho_{t,T}$ to the domain[4] $\left[\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\right]_{s=t}^{T}$ yields constant functions, that is,

$$\varrho_{t,T} : \left[\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\right]_{s=t}^{T} \to \bar{\mathbb{R}} \; , \tag{30b}$$

A *Markov dynamic uncertainty criterion* is a family $\left\{\left\{\varrho_{t,T}^{x_t}\right\}_{x_t\in\mathbb{X}_t}\right\}_{t=0}^{T}$, such that $\left\{\varrho_{t,T}^{x_t}\right\}_{t=0}^{T}$ is a dynamic uncertainty criterion, for all sequence $\left\{x_t\right\}_{t=0}^{T}$ of states, where $x_t \in \mathbb{X}_t$, for all $t \in [\![0,T]\!]$.

We relate dynamic uncertainty criteria and optimization problems as follows.

Definition 5. Given a *Markov dynamic uncertainty criterion* $\left\{\left\{\varrho_{t,T}^{x_t}\right\}_{x_t\in\mathbb{X}_t}\right\}_{t=0}^{T}$, we define a *Markov optimization problem* as the following sequence of families of optimization problems:

$$(\mathfrak{P}_t)(x) \qquad \min_{\pi\in\Pi^{\mathrm{ad}}} \varrho_{t,T}^{x}\Big(\left\{J_{t,s}^{x,\pi}\right\}_{s=t}^{T}\Big) \; , \;\; \forall t \in [\![0,T]\!] \; , \;\; \forall x \in \mathbb{X}_t \; . \tag{31}$$

Each Problem (31) is indeed well defined by (30b), because $\left\{J_{t,s}^{x,\pi}\right\}_{s=t}^{T} \in \left[\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\right]_{s=t}^{T}$ by (28).

[4] Where $\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})$ is naturally identified as a subset of $\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})$.

2.2.2. Dynamic Risk Measures in a Nutshell

We establish a parallel between uncertainty criteria and risk measures. For this purpose, when needed, we implicitely suppose that each uncertainty set $\mathbb{W}_t$ is endowed with a σ-algebra $\mathcal{W}_t$, so that the set $\mathbb{W}_{[0:T]}$ of scenarios is naturally equipped with the filtration

$$\mathfrak{F}_t = \mathcal{W}_0 \otimes \cdots \otimes \mathcal{W}_t \otimes \{\emptyset, \mathbb{W}_{t+1}\} \otimes \cdots \otimes \{\emptyset, \mathbb{W}_T\} \ , \ \ \forall t \in [\![0, T]\!] \ . \tag{32}$$

Then, we make the correspondence between (see also the correspondence Table 1)

- the measurable space $(\mathbb{W}_{[0:T]}, \mathfrak{F}_T)$ and the measurable space $(\Omega, \mathfrak{F})$ in §2.2.2,
- the set $\mathcal{F}\big(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}\big)$ of functions and a set $\mathcal{L}_t$ of random variables that are $\mathfrak{F}_t$-measurable in §2.2.2,
- the set $\big[\mathcal{F}(\mathbb{W}_{[s:T]}; \bar{\mathbb{R}})\big]_{s=t}^{T}$ and a set $\mathcal{L}_{t,T}$ of adapted processes, as in (35) in §2.2.2.

Notice that, when the σ-algebra $\mathcal{W}_t$ is the complete σ-algebra made of all subsets of $\mathbb{W}_t$, $\mathcal{F}\big(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}\big)$ is exactly the space of random variables that are $\mathfrak{F}_t$-measurable.

We follow the seminal work [26], as well as [27, 28], for recalls about risk measures.

Static Risk Measures. Let $\big(\Omega, \mathfrak{F}\big)$ be a measurable space. Let $\mathcal{L}$ be a vector space of measurable functions taking values in $\mathbb{R}$ (for example, $\mathcal{L} = \mathrm{L}^p\big(\Omega, \mathfrak{F}, \mathbb{P}; \mathbb{R}\big)$). We endow the space $\mathcal{L}$ with the following partial order:

$$\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{L}, \qquad \boldsymbol{X} \leq \boldsymbol{Y} \quad \iff \quad \forall \omega \in \Omega \ , \ \ \boldsymbol{X}(\omega) \leq \boldsymbol{Y}(\omega) \ .$$

Definition 6. A *risk measure* (with domain $\mathcal{L}$) is a mapping $\rho : \mathcal{L} \to \mathbb{R}$.

A *convex risk measure* is a mapping $\rho : \mathcal{L} \to \mathbb{R}$ displaying the following properties:

- *Convexity:* $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{L}$, $\forall t \in [0, 1]$, $\qquad \rho\big(t\boldsymbol{X} + (1-t)\boldsymbol{Y}\big) \leq t\rho\big(\boldsymbol{X}\big) + (1-t)\rho\big(\boldsymbol{Y}\big)$,
- *Monotonicity:* if $\boldsymbol{Y} \geq \boldsymbol{X}$, then $\rho\big(\boldsymbol{Y}\big) \geq \rho\big(\boldsymbol{X}\big)$,
- *Translation equivariance:* $\forall c \in \mathbb{R}$, $\forall \boldsymbol{X} \in \mathcal{L}$, $\qquad \rho(c + \boldsymbol{X}) = c + \rho(\boldsymbol{X})$.

A *coherent risk measure* is a convex risk measure $\rho : \mathcal{L} \to \mathbb{R}$ with the following additional property:

- *Positive homogeneity:* $\forall t \geq 0$, $\forall \boldsymbol{X} \in \mathcal{L}$, $\qquad \rho(t\boldsymbol{X}) = t\rho(\boldsymbol{X})$.

Let $\mathcal{P}$ be a set of probabilities on $\big(\Omega, \mathfrak{F}\big)$ and let Υ be a function mapping the space of probabilities on $\big(\Omega, \mathfrak{F}\big)$ onto $\mathbb{R}$. The functional defined by

$$\rho(\boldsymbol{X}) = \sup_{\mathbb{P} \in \mathcal{P}} \big\{ \mathbb{E}_{\mathbb{P}}\big[\boldsymbol{X}\big] - \Upsilon(\mathbb{P}) \big\} \tag{33}$$

is a convex risk measure on a proper domain $\mathcal{L}$ (for instance, the bounded functions over Ω). The expression

$$\rho(\boldsymbol{X}) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[\boldsymbol{X}\big] \tag{34}$$

defines a coherent risk measure.

Under proper technical assumptions, it can be shown that any convex or coherent risk measure can be represented by the above expressions.

Conditional Risk Mappings. We present the *conditional risk mappings* as defined in [28], extending the work of [29].

Let $\big(\Omega, \mathfrak{F}\big)$ be a measurable space, $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \mathfrak{F}$ be two σ-algebras, and $\mathcal{L}_1 \subset \mathcal{L}_2$ be two vector spaces of functions $\Omega \to \mathbb{R}$ that are measurable with respect to $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively.

Definition 7. A *conditional risk mapping* is a mapping $\rho : \mathcal{L}_2 \to \mathcal{L}_1$.

A *convex conditional risk mapping* $\rho : \mathcal{L}_2 \to \mathcal{L}_1$ has the following properties:

- *Convexity:* $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{L}_2$, $\forall t \in [0,1]$, $\quad \rho\big(t\boldsymbol{X} + (1-t)\boldsymbol{Y}\big) \leq t\rho\big(\boldsymbol{X}\big) + (1-t)\rho\big(\boldsymbol{Y}\big)$,
- *Monotonicity:* if $\boldsymbol{Y} \geq \boldsymbol{X}$, then $\rho\big(\boldsymbol{Y}\big) \geq \rho\big(\boldsymbol{X}\big)$,
- *Translation equivariance:* $\forall c \in \mathcal{L}_1$, $\forall \boldsymbol{X} \in \mathcal{L}_2$, $\quad \rho(c + \boldsymbol{X}) = c + \rho(\boldsymbol{X})$.

Conditional and Dynamic Risk Measures. We follow [23, Section 3]. Let $\big(\Omega, \mathfrak{F}\big)$ be a measurable space, with a filtration $\mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_T \subset \mathfrak{F}$, and $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_T$ be vector spaces of functions $\Omega \to \mathbb{R}$ that are measurable with respect to $\mathfrak{F}_1$, $\ldots$, $\mathfrak{F}_T$, respectively. We set

$$\mathcal{L}_{t,T} = \mathcal{L}_t \times \cdots \times \mathcal{L}_T \ , \ \ \forall t \in [\![0, T]\!] \ . \tag{35}$$

An element $\{\boldsymbol{A}_s\}_0^T$ of $\mathcal{L}_{t,T}$ is an *adapted process* since every $\boldsymbol{A}_s \in \mathcal{L}_s$ is $\mathfrak{F}_s$-measurable. Conditional and dynamic risk measures have adapted processes as arguments, to the difference of risk measures that take random variables as arguments.

Definition 8. Let $t \in [\![0, T]\!]$. A *one-step conditional risk mapping* is a conditional risk mapping $\rho_t : \mathcal{L}_{t+1} \to \mathcal{L}_t$. A *conditional risk measure* is a mapping $\rho_{t,T} : \mathcal{L}_{t,T} \mapsto \mathcal{L}_t$.

A *dynamic risk measure* is a sequence $\big\{\rho_{t,T}\big\}_{t=0}^T$ of conditional risk measures.

Dynamic uncertainty criteria $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 4 correspond to *dynamic risk measures.*

Remark 4. *A conditional risk measure* $\rho_{t,T} : \mathcal{L}_{t,T} \mapsto \mathcal{L}_t$ *is said to be* monotonous[5] *if, for all* $\{\underline{\boldsymbol{A}}_s\}_{s=t}^{T}$ *and* $\{\overline{\boldsymbol{A}}_s\}_{s=t}^{T}$ *in* $\mathcal{L}_{t,T}$*, we have*

$$\forall s \in [\![t, T]\!] \ , \ \ \underline{\boldsymbol{A}}_s \leq \overline{\boldsymbol{A}}_s \Longrightarrow \rho_{t,T}\big(\{\underline{\boldsymbol{A}}_s\}_{s=t}^{T}\big) \leq \rho_{t,T}\big(\{\overline{\boldsymbol{A}}_s\}_{s=t}^{T}\big) \ . \tag{36}$$

Markov Risk Measures. In [23], *Markov risk measures* are defined with respect to a given controlled Markov process. We adapt this definition to the setting developed in the Introduction, and we consider the control stochastic dynamical system (3b)

$$\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t) \ ,$$

where $\{\boldsymbol{W}_t\}_0^T$ is a sequence of independent random variables. Then, for all policy π, when $\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t)$ we obtain a Markov process $\{\boldsymbol{X}_t\}_{t\in[\![0,T]\!]}$, where $\boldsymbol{X}_t = X_{0,t}^{x_0,\pi}\big(\{\boldsymbol{W}_s\}_0^{t-1}\big)$ is given by the flow (23).

Let $\{\mathfrak{F}_t\}_{t=0}^{T}$ be the filtration defined by $\mathfrak{F}_t = \sigma(\{\boldsymbol{W}_s\}_0^t)$. For any $t \in [\![0, T]\!]$, let $\mathcal{V}_t$ be a set of functions mapping $\mathbb{X}_t$ into $\mathbb{R}$ such that we have $v\big(\boldsymbol{X}_{0,t}^{x_0,\pi}\big) \in \mathcal{L}_t$, for all policy $\pi \in \Pi^{\mathrm{ad}}$.

Definition 9. A one-step conditional risk measure $\rho_{t-1} : \mathcal{L}_t \to \mathcal{L}_{t-1}$ is a *Markov risk measure* with respect to the control stochastic dynamical system (3b) if there exists a function $\Psi_t : \mathcal{V}_{t+1} \times \mathbb{X}_t \times \mathbb{U}_t \to \mathbb{R}$, such that, for any policy $\pi \in \Pi^{\mathrm{ad}}$, and any function $v \in \mathcal{V}_{t+1}$, we have

$$\begin{aligned}
&\rho_{t-1}\bigg(\{\boldsymbol{W}_s\}_0^t \mapsto v\Big(\boldsymbol{X}_{0,t+1}^{x_0,\pi}\big(\{\boldsymbol{W}_s\}_0^t\big)\Big)\bigg) \\
=&\Psi_t\bigg(v, \boldsymbol{X}_{0,t}^{x_0,\pi}\big(\{\boldsymbol{W}_s\}_0^{t-1}\big), \pi_t\Big(\boldsymbol{X}_{0,t}^{x_0,\pi}\big(\{\boldsymbol{W}_s\}_0^{t-1}\big)\Big)\bigg) \ .
\end{aligned} \tag{37}$$

A Markov risk measure is said to be coherent (resp. convex) if, for any state $x \in \mathbb{X}_t$, any control $u \in \mathbb{U}_t$, the function

$$v \mapsto \Psi_t\big(v, x, u\big) \ , \tag{38}$$

is a coherent (resp convex) risk measure on $\mathcal{V}_{t+1}$ (equipped with a proper σ-algebra).

Dynamic Markov uncertainty criteria $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 4 correspond to *Markov risk measures.*

Correspondence Table.

[5] In [23, Section 3], a conditional risk measure is necessarily monotonous, by definition.

Risk Measures		Uncertainty Criteria	
measurable space	$(\Omega, \mathfrak{F})$	$(\mathbb{W}_{[0:T]}, \mathfrak{F}_T)$	measurable space
$\mathfrak{F}_t$-measurable		$\mathcal{F}\big(\mathbb{W}_{[0:t]}; \mathbb{R}\big)$	
adapted processes	$\mathcal{L}_{0,T}$	$\big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})\big]_{s=0}^{T}$	adapted uncertainty processes
dynamic risk measure	$\{\rho_{t,T}\}_{t=0}^{T}$	$\{\varrho_{t,T}\}_{t=0}^{T}$	dynamic uncertainty criteria
Markov dynamic risk measure	$\Big\{\big\{\rho_{t,T}^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$	$\Big\{\big\{\varrho_{t,T}^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$	Markov dynamic uncertainty criterion

Table 1: Correspondence Table

Time-Consistency for Dynamic Risk Measures. The literature on risk measures has introduced a notion of *time-consistency for dynamic risk measures*, that we recall here (see [30, 29, 9]).

Definition 10. A dynamic risk measure $\{\rho_{t,T}\}_{t=0}^{T}$, where $\rho_{t,T} : \mathcal{L}_{t,T} \mapsto \mathcal{L}_t$, is said to be *time-consistent* if, for any couples of times $0 \leq \underline{t} < \overline{t} \leq T$, the following property holds true. If two adapted stochastic processes $\{\underline{\boldsymbol{A}}_s\}_0^T$ and $\{\overline{\boldsymbol{A}}_s\}_0^T$ in $\mathcal{L}_{0,T}$ satisfy

$$\underline{\boldsymbol{A}}_s = \overline{\boldsymbol{A}}_s \; , \qquad \forall s \in [\![\underline{t}, \overline{t}-1]\!] \; , \tag{39a}$$

$$\rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \leq \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \; , \tag{39b}$$

then we have:

$$\rho_{\underline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\underline{t}}^T\big) \leq \rho_{\underline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\underline{t}}^T\big) \; . \tag{39c}$$

Remark 5. *In [23], the equality* (39a) *is replaced by the inequality*

$$\underline{\boldsymbol{A}}_s \leq \overline{\boldsymbol{A}}_s \; , \;\; \forall s \in [\![\underline{t}, \overline{t}]\!] \; . \tag{39d}$$

Depending whether we choose (39a) *or* (39d) *as assumption to define a time-consistent dynamic risk measure, we have to adapt or not an assumption in Theorem 9 (see Remark 10).*

2.3. Definitions of Time-Consistency

With the formalism of §2.2.1, we give a definition of time-consistency for Markov optimization problems in §2.3.1, and for Markov dynamic uncertainty criteria in §2.3.2.

2.3.1. Time-Consistency for Markov Optimization Problems

With the formalism of §2.2.1, we here give a definition of time-consistency for Markov optimization problems. We refer the reader to Definition 5 for the terminology.

Consider the Markov optimization problem $\big\{\{(\mathfrak{P}_t)(x)\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in (31). For the clarity of exposition, suppose for a moment that any optimization Problem $(\mathfrak{P}_t)(x)$ has a unique solution, that we denote $\pi^{t,x} = \{{\pi_s}^{t,x}\}_{s=t}^{T-1} \in \Pi_t^{\mathrm{ad}}$. Consider $0 \leq \underline{t} < \overline{t} \leq T$. Suppose that, starting from the state $\underline{x}$ at time $\underline{t}$, the flow (23) drives you to

$$\overline{x} = X_{\underline{t},\overline{t}}^{\underline{x},\pi}(w) \ , \quad \pi = \pi^{\underline{t},\underline{x}} \tag{40}$$

at time $\overline{t}$, along the scenario $w \in \mathbb{W}_{[0:T]}$ and adopting the optimal policy $\pi^{\underline{t},\underline{x}} \in \Pi_{\underline{t}}^{\mathrm{ad}}$. Arrived at $\overline{x}$, you solve $(\mathfrak{P}_{\overline{t}})(\overline{x})$ and get the optimal policy $\pi^{\overline{t},\overline{x}} = \{{\pi_s}^{\overline{t},\overline{x}}\}_{s=\overline{t}}^{T-1} \in \Pi_{\overline{t}}^{\mathrm{ad}}$. Time-consistency holds true when

$$\forall s \geq \overline{t}, \quad {\pi_s}^{\overline{t},\overline{x}} = {\pi_s}^{\underline{t},\underline{x}} \ , \tag{41}$$

that is, when the "new" optimal policy, obtained by solving $(\mathfrak{P}_{\overline{t}})(\overline{x})$, coincides, after time $\overline{t}$, with the "old" optimal policy, obtained by solving $(\mathfrak{P}_{\underline{t}})(\underline{x})$. In other words, you "stick to your plans" (here, a plan is a policy) and do not reconsider your policy whenever you stop along an optimal path and optimize ahead from this stop point.

To account for non-uniqueness of optimal policies, we propose the following formal definition.

Definition 11. For any policy $\pi \in \Pi$, suppose given a Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$. We say that the Markov optimization problem

$$(\mathfrak{P}_t)(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ . \tag{42}$$

is *time-consistent* if, for any couple of times $\underline{t} \leq \overline{t}$ in $[\![0,T]\!]$ and any state $\underline{x} \in \mathbb{X}_{\underline{t}}$, the following property holds: there exists a policy $\pi^\sharp = \{\pi_s^\sharp\}_{s=\underline{t}}^{T-1} \in \Pi_{\underline{t}}^{\mathrm{ad}}$ such that

- $\{\pi_s^\sharp\}_{s=\underline{t}}^{T-1}$ is optimal for Problem $\mathfrak{P}_{\underline{t}}(\underline{x})$;
- the tail policy $\{\pi_s^\sharp\}_{s=\overline{t}}^{T-1}$ is optimal for Problem $\mathfrak{P}_{\overline{t}}(\overline{x})$, where $\overline{x} \in \mathbb{X}_{\overline{t}}$ is any state achieved by the flow $X_{\underline{t},\overline{t}}^{\underline{x},\pi^\sharp}$ in (23).

We stress that the above definition of time-consistency of a sequence of families of optimization problems is contingent on the state x and on the dynamics $\big\{f_t\big\}_0^{T-1}$ by the flow (23). In particular, we assume that, at each time step, the control is taken only in function of the state: this defines the class of solutions as policies that are feedbacks of the state x.

2.3.2. Time-Consistency for Markov Dynamic Uncertainty Criteria

We provide a definition of time-consistency for Markov dynamic uncertainty criteria, inspired by the definitions of time-consistency for, on the one hand, dynamic risk measures (recalled in §2.2.2) and, on the other hand, Markov optimization problems. We refer the reader to Definition 4 for the terminology.

Definition 12. The Markov dynamic uncertainty criterion $\{\{\varrho^{x_t}_{t,T}\}_{x_t \in \mathbb{X}_t}\}_{t=0}^{T}$ is said to be *time-consistent* if, for any couple of times $0 \leq \underline{t} < \overline{t} \leq T$, the following property holds true.

If two adapted uncertainty processes $\{\underline{A}_s\}_0^T$ and $\{\overline{A}_s\}_0^T$, satisfy

$$\underline{A}_s = \overline{A}_s , \qquad \forall s \in [\![\underline{t}, \overline{t}]\!] , \tag{43a}$$

$$\rho^{\overline{x}}_{\overline{t},T}\big(\{\underline{A}_s\}_{\overline{t}}^T\big) \leq \rho^{\overline{x}}_{\overline{t},T}\big(\{\overline{A}_s\}_{\overline{t}}^T\big) , \qquad \forall \overline{x} \in \mathbb{X}_{\overline{t}} , \tag{43b}$$

then we have:

$$\rho^{\underline{x}}_{\underline{t},T}\big(\{\underline{A}_s\}_{\underline{t}}^T\big) \leq \rho^{\underline{x}}_{\underline{t},T}\big(\{\overline{A}_s\}_{\underline{t}}^T\big) , \qquad \forall \underline{x} \in \mathbb{X}_{\underline{t}} . \tag{43c}$$

This Definition 12 of time-consistency is quite different from Definition 11. Indeed, if the latter looks after consistency between solutions to intertemporal optimization problems, the former is a monotonicity property. Several authors establish connections between these two definitions [31, 32, 23, 33] for case specific problems. In the following §3, we provide what we think is one of the most systematic connections between time-consistency for Markov dynamic uncertainty criteria and time-consistency for intertemporal optimization problems.

3. Proving Joint Time-Consistency

In §3.1, we introduce the notions of time and uncertainty-aggregators, define their composition, and outline the general four ways to craft a dynamic uncertainty criterion from one-step aggregators. In §3.2, we present two ways to craft a nested dynamic uncertainty criterion; for each of them, we provide sufficient monotonicity assumptions on one-step aggregators that ensure time-consistency and the existence of a DPE. In §3.3, we introduce two commutation properties, that will be the key ingredients for time-consistency and for the existence of a DPE in non-nested cases. In §3.4, we present two ways to craft a non-nested dynamic uncertainty criterion; for each of them, we provide sufficient monotonicity and commutation assumptions on one-step aggregators that ensure time-consistency and the existence of a DPE.

3.1. Aggregators and their Composition

We introduce the notions of time and uncertainty-aggregators, define their composition, and outline the general four ways to craft a dynamic uncertainty criterion from one-step aggregators.

3.1.1. One-Step Time-Aggregators and their Composition

Time preferences are reflected in how streams of costs — elements of $\bar{\mathbb{R}}^{T+1}$, like $\{J_{0,t}^{x,\pi}(w)\}_{t=0}^{T}$ introduced in Definition 3 — are aggregated with respect to time thanks to a function $\Phi : \bar{\mathbb{R}}^{T+1} \to \bar{\mathbb{R}}$, called *multiple-step time-aggregator*. Commonly, multiple-step time-aggregators are built progressively backward. In §1.1, the multiple-step time-aggregator is the time-separable and additive $\Phi\big\{c_s\big\}_{s=0}^{T} = \sum_{s=0}^{T} c_s$, obtained as the initial value of the backward induction $\sum_{s=t}^{T} c_s = (\sum_{s=t+1}^{T} c_s) + c_t$; the time-separable and multiplicative aggregator $\Phi\big\{c_s\big\}_{s=0}^{T} = \prod_{s=0}^{T} c_s$ is the initial value of the backward induction $\prod_{s=t}^{T} c_s = (\prod_{s=t+1}^{T} c_s)c_t$. A multiple-step time-aggregator aggregates the $T+1$ costs $\{J_{0,t}^{x,\pi}(w)\}_{t=0}^{T}$, whereas a one-step time-aggregator aggregates two costs, the current one and the "cost-to-go" (as in [13]).

Definition 13. A *multiple-step time-aggregator* is a function mapping $\bar{\mathbb{R}}^k$ into $\bar{\mathbb{R}}$, where $k \geq 2$. When $k = 2$, we call *one-step time-aggregator* a function mapping $\bar{\mathbb{R}}^2$ into $\bar{\mathbb{R}}$.

A one-step time-aggregator is said to be *non-decreasing* if it is non-decreasing in its second variable.

We define the composition of time-aggregators as follows.

Definition 14. Let $\Phi^1 : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ be a one-step time-aggregator and $\Phi^k : \bar{\mathbb{R}}^k \to \bar{\mathbb{R}}$ be a multiple-step time-aggregator. We define $\Phi^1 \odot \Phi^k : \bar{\mathbb{R}}^{k+1} \to \bar{\mathbb{R}}$ by

$$\Big(\Phi^1 \odot \Phi^k\Big)\big\{c_1, c_2, \ldots, c_{k+1}\big\} = \Phi^1\Big\{c_1, \Phi^k\big\{c_2, \ldots, c_{k+1}\big\}\Big\} . \tag{44}$$

Quite naturaly, we define the composition of sequences of one-step time-aggregators as follows.

Definition 15. Consider a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators $\Phi_t : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}$, for $t \in [\![0, T-1]\!]$. For all $t \in [\![0, T-1]\!]$, we define the composition $\bigodot_{s=t}^{T-1} \Phi_s$ as the multiple-step time-aggregator from $\bar{\mathbb{R}}^{T+1-t}$ towards $\bar{\mathbb{R}}$, inductively given by

$$\bigodot_{t=T-1}^{T-1} \Phi_t = \Phi_{T-1} \text{ and } \Big(\bigodot_{s=t}^{T-1} \Phi_s\Big) = \Phi_t \odot \Big(\bigodot_{s=t+1}^{T-1} \Phi_s\Big) . \tag{45a}$$

That is, for all sequence $c_{[t:T]}$ where $c_s \in \bar{\mathbb{R}}$, we have:

$$\Big(\bigodot_{s=t}^{T-1} \Phi_s\Big)\big(c_{[t:T]}\big) = \Phi_t\Big\{c_t, \Big(\bigodot_{s=t+1}^{T-1} \Phi_s\Big)\big(c_{[t+1:T]}\big)\Big\} . \tag{45b}$$

Example 6. *Consider the sequence* $\big\{\Phi_t\big\}_{t=0}^{T-1}$ *of one-step time-aggregators given by*

$$\Phi_t\big\{c_t, c_{t+1}\big\} \quad = \quad \alpha_t(c_t) + \beta_t(c_t)c_{t+1} \ , \ \ \forall t \in [\![0, T-1]\!] \ , \tag{46}$$

where $(\alpha_t)_{t\in[\![0,T-1]\!]}$ *and* $(\beta_t)_{t\in[\![0,T-1]\!]}$ *are sequences of functions, each mapping* $\bar{\mathbb{R}}$ *into* $\mathbb{R}$. *We have*

$$\Big(\mathop{\odot}_{s=t}^{T-1} \Phi_s\Big)\big\{c_s\big\}_t^T = \sum_{s=t}^{T} \Big(\alpha_s\big(c_s\big) \prod_{r=t}^{s-1} \beta_r\big(c_r\big)\Big) \ , \ \ \forall t \in [\![0, T-1]\!] \ , \tag{47}$$

with the convention that $\alpha_T(c_T) = c_T$.

Example 7. *Consider the one-step aggregators*

$$\Phi\{c_1, c_2\} = c_1 + c_2 \ , \ \ \Psi\{c_1, c_2\} = c_1 c_2 \ .$$

The first one Φ *corresponds to the sum, as in* (3)*; the second one* Ψ *corresponds to the product, as in* (11)*. As an illustration, we form four compositions (multiple-step time-aggregators):*

$$\begin{aligned}
\Phi \odot \Phi\{c_1, c_2, c_3\} &= \Phi\big\{c_1, \Phi\{c_2, c_3\}\big\} = c_1 + c_2 + c_3 \ , \\
\Psi \odot \Psi\{c_1, c_2, c_3\} &= \Psi\big\{c_1, \Psi\{c_2, c_3\}\big\} = c_1 c_2 c_3 \ , \\
\Phi \odot \Psi\{c_1, c_2, c_3\} &= \Phi\big\{c_1, \Psi\{c_2, c_3\}\big\} = c_1 + c_2 c_3 \ , \\
\Psi \odot \Phi\{c_1, c_2, c_3\} &= \Psi\big\{c_1, \Phi\{c_2, c_3\}\big\} = c_1(c_2 + c_3).
\end{aligned}$$

We extend the composition $\Big(\mathop{\odot}_{s=t}^{T-1} \Phi_s\Big) : \bar{\mathbb{R}}^{T+1-t} \to \bar{\mathbb{R}}$ into a mapping (48) as follows.

Definition 16. Consider a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators, for $t \in [\![0, T-1]\!]$. For $t \in [\![0, T-1]\!]$, we define the composition[6] $\Big\langle \mathop{\odot}_{s=t}^{T-1} \Phi_s \Big\rangle$ as a mapping

$$\Big\langle \mathop{\odot}_{s=t}^{T-1} \Phi_s \Big\rangle : \Big(\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})\Big)^{T-t+1} \to \mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}) \tag{48}$$

by, for any $\{A\}_t^T \in \Big(\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})\Big)^{T-t+1}$,

$$\Big(\Big\langle \mathop{\odot}_{s=t}^{T-1} \Phi_s \Big\rangle \Big(\{A\}_t^T\Big)\Big)\big(w\big) = \Big(\mathop{\odot}_{s=t}^{T-1} \Phi_s\Big)\big(\{A_t\big(w\big)\}_t^T\big) \ , \ \ \forall w \in \mathbb{W}_{[0:T]} \ . \tag{49}$$

In other words, we simply plug the values $\{A_t\big(w\big)\}_t^T$ into $\Big(\mathop{\odot}_{s=t}^{T-1} \Phi_s\Big)$.

[6] We will consistently use the symbol $\Big\langle \quad \Big\rangle$ to denote a mapping with image a set of functions.

3.1.2. One-Step Uncertainty-Aggregators and their Composition

As with time, risk or uncertainty preferences are materialized by a function $\mathbb{G} : \mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}) \to \bar{\mathbb{R}}$, called *multiple-step uncertainty-aggregator*. A multiple-step aggregator is usually defined on a subset $\mathbb{F}$ of $\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})$ (for example the measurable and integrable functions), and then extended to $\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})$ by setting $\mathbb{G}[A] = +\infty$ for any function $A \notin \mathbb{F}$. Indeed, as we are interested in minimizing $\mathbb{G}$, being not defined or equal to $+\infty$ amount to the same result.

In the first part of §1.1, the multiple-step uncertainty-aggregator is the extended expectation with respect to the probability $\mathbb{P}$; still denoted by $\mathbb{E}_{\mathbb{P}}$, it is defined as the usual expectation if the operand is measurable and integrable, and as $+\infty$ otherwise. In the second part of §1.1, the multiple-step uncertainty-aggregator is the fear operator, namely the supremum $\sup_{w \in \mathbb{W}_{[0:T]}}$ over scenarios in $\mathbb{W}_{[0:T]}$.

Definition 17. Let $t \in [\![0, T]\!]$ and $s \in [\![t, T]\!]$. A $[t\!:\!s]$*-multiple-step uncertainty-aggregator* is a mapping[7] $\mathbb{G}^{[t:s]}$ from $\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}})$ into $\bar{\mathbb{R}}$. When $t = s$, we call $\mathbb{G}^{[t:t]}$ a *t-one-step uncertainty-aggregator*.

A $[t\!:\!s]$-multiple-step uncertainty-aggregator is said to be *non-decreasing* if, for any functions[8] $\underline{D}_t$ and $\overline{D}_t$ in $\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}})$, we have

$$\big(\forall w_{[t:s]} \in \mathbb{W}_{[t:s]} \ , \ \ \underline{D}_t\big(w_{[t:s]}\big) \leq \overline{D}_t\big(w_{[t:s]}\big)\big) \Longrightarrow \mathbb{G}^{[t:s]}\big[\underline{D}_t\big] \leq \mathbb{G}^{[t:s]}\big[\overline{D}_t\big] \ .$$

Definition 18. Let $t \in [\![1, T]\!]$ and $s \in [\![t, T]\!]$. To a $[t\!:\!s]$-multiple-step uncertainty-aggregator $\mathbb{G}^{[t:s]}$, we attach a mapping[9]

$$\Big\langle \mathbb{G}^{[t:s]} \Big\rangle : \mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}) \to \mathcal{F}(\mathbb{W}_{[0:t-1]}; \bar{\mathbb{R}}) \ , \tag{50a}$$

obtained by freezing the first variables as follows. For any $A : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$, and any $w_{[0:s]} \in \mathbb{W}_{[0:s]}$, we set

$$\Big(\Big\langle \mathbb{G}^{[t:s]} \Big\rangle \left[A\right] \Big) \big(w_{[0:t-1]}\big) = \mathbb{G}^{[t:s]} \Big[w_{[t:s]} \mapsto A\big(w_{[0:t-1]}, w_{[t:s]}\big) \Big] \ . \tag{50b}$$

Multiple-step uncertainty-aggregators are commonly built progressively backward: in §1.1, the expectation operator $\mathbb{E}_{\mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_T}$ is the initial value of the induction $\mathbb{E}_{\mathbb{P}_t \otimes \cdots \otimes \mathbb{P}_T} = \mathbb{E}_{\mathbb{P}_t} \mathbb{E}_{\mathbb{P}_{t+1} \otimes \cdots \otimes \mathbb{P}_T}$; the fear operator $\sup_{w \in \mathbb{W}_{[0:T]}}$ is the initial value of the induction $\sup_{w \in \mathbb{W}_{[t:T]}} = \sup_{w_t \in \mathbb{W}_t} \sup_{w \in \mathbb{W}_{[t+1:T]}}$.

We define the composition of uncertainty-aggregators as follows.

---

[7] The superscript notation indicates that the domain of the mapping $\mathbb{G}^{[t:s]}$ is $\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}})$ (not to be confused with $\mathbb{G}_{[t:s]} = \big\{\mathbb{G}_r\big\}_{r=t}^{s}$).

[8] We will consistently use the symbol D to denote a function in $\mathcal{F}\big(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}\big)$, that is, $D : \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}}$.

[9] See Footnote 6 about the notation $\langle \quad \rangle$.

Definition 19. Let $t \in \llbracket 0,T \rrbracket$ and $s \in \llbracket t+1,T \rrbracket$. Let $\mathbb{G}^{[t:t]} : \mathcal{F}(\mathbb{W}_t;\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$ be a t-one-step uncertainty-aggregator, and $\mathbb{G}^{[t+1:s]} : \mathcal{F}(\mathbb{W}_{[t+1:s]};\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$ be a $[t+1:s]$-multiple-step uncertainty-aggregator. We define the $[t:s]$-multiple-step uncertainty-aggregator $\mathbb{G}^{[t:t]} \boxdot \mathbb{G}^{[t:s]}$ by

$$\Big(\mathbb{G}^{[t:t]} \boxdot \mathbb{G}^{[t:s]}\Big)\big[A_t\big] = \mathbb{G}^{[t:t]}\Big[w_t \mapsto \mathbb{G}^{[t+1:s]}\big[w_{[t+1:s]} \mapsto A_t\big(w_t, w_{[t+1:s]}\big)\big]\Big] \;, \tag{51}$$

for all function $A_t \in \mathcal{F}\big(\mathbb{W}_{[t:s]};\bar{\mathbb{R}}\big)$.

Quite naturaly, we define the composition of sequences of one-step uncertainty-aggregators as follows.

Definition 20. We say that a sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators is a *chained sequence* if $\mathbb{G}_t$ is a t-one-step uncertainty-aggregator, for all $t \in \llbracket 0,T \rrbracket$.

Consider a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators. For $t \in \llbracket 0,T \rrbracket$, we define the composition $\mathop{\boxdot}\limits_{s=t}^{T} \mathbb{G}_s$ as the $[t:T]$-multiple-step uncertainty-aggregator

$$\mathop{\boxdot}_{s=t}^{T} \mathbb{G}_s : \mathcal{F}\big(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}\big) \to \bar{\mathbb{R}} \;, \tag{52}$$

inductively given by

$$\mathop{\boxdot}_{s=T}^{T} \mathbb{G}_s = \mathbb{G}_T \text{ and } \Big(\mathop{\boxdot}_{s=t}^{T} \mathbb{G}_s\Big) = \mathbb{G}_t \boxdot \Big(\mathop{\boxdot}_{s=t+1}^{T} \mathbb{G}_s\Big) \;. \tag{53a}$$

That is, for all function $B_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}\big)$, we have:

$$\Big(\mathop{\boxdot}_{s=t}^{T} \mathbb{G}_s\Big)\big[B_t\big] = \mathbb{G}_t\Big[w_t \mapsto \Big(\mathop{\boxdot}_{s=t+1}^{T} \mathbb{G}_s\Big)\big[w_{[t+1:T]} \mapsto B_t\big(w_t, w_{[t+1:T]}\big)\big]\Big] \;. \tag{53b}$$

3.1.3. Crafting Dynamic Uncertainty Criteria from Aggregators

We outline four ways to craft a dynamic uncertainty criterion from aggregators. Let $A_{[0:T]} = \big\{A_s\big\}_{s=0}^{T}$ denote an arbitrary adapted uncertainty process (that is, $A_s : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$, as in Definition 1).

Non Nested Dynamic Uncertainty Criteria. The two following ways to craft a dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$ display a natural economic interpretation in term of preferences over streams of uncertain costs like $A_{[0:T]}$. They mix time and uncertainty preferences, either first with respect to uncertainty then with respect to time (UT) or first with respect to time, then with respect to uncertainty (TU). However, they are not directly amenable to a DPE.

TU, or time, then uncertainty. Let $t \in \llbracket 0,T \rrbracket$ be fixed.

- First, we aggregate $A_{[t:T]}$ with respect to time by means of a multiple-step time-aggregator Φ^t from $\bar{\mathbb{R}}^{T-t+1}$ towards $\bar{\mathbb{R}}$, and we obtain $\Phi^t\big(A_{[t:T]}\big)$.
- Second, we aggregate $\Phi^t\big(A_{[t:T]}\big)$ with respect to uncertainty by means of a multiple-step uncertainty-aggregator $\mathbb{G}^{[t:T]}$, and we obtain

$$\varrho_{t,T}\big(A_{[t:T]}\big) = \Big\langle \mathbb{G}^{[t:T]} \Big\rangle \Big[\Phi^t\big(A_{[t:T]}\big) \Big] \;. \tag{54}$$

All the examples in §1.1 belong to this TU class, and some in §1.2.

UT, or uncertainty, then time.

- First, we aggregate $A_{[t:T]}$ with respect to uncertainty by means of a sequence $\big[{\mathbb{G}_s}^{[t:s]}\big]_{s=t}^{T}$ of multiple-step time-aggregators ${\mathbb{G}_t}^{[t:s]} : \mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}) \to \bar{\mathbb{R}}$, and we obtain a sequence $\Big\{ \Big\langle {\mathbb{G}_s}^{[t:s]} \Big\rangle \big[A_s\big] \Big\}_{s=t}^{T}$.
- Second, we aggregate $\Big\{ \Big\langle {\mathbb{G}_s}^{[t:s]} \Big\rangle \big[A_s\big] \Big\}_{s=t}^{T}$ by means of a multiple-step time-aggregator Φ^t from $\bar{\mathbb{R}}^{T-t+1}$ towards $\bar{\mathbb{R}}$, and we obtain

$$\varrho_{t,T}\big(A_{[t:T]}\big) = \Phi^t\bigg(\Big\{ \Big\langle {\mathbb{G}_s}^{[t:t]} \Big\rangle \big[A_s\big] \Big\}_{s=t}^{T} \bigg) \;. \tag{55}$$

Some examples in §1.2 belong to this UT class.

Nested Dynamic Uncertainty Criteria. The two following ways to craft a dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$ do not display a natural economic interpretation in term of preferences [34], but they are directly amenable to a DPE. Indeed, they are produced by a backward induction, nesting uncertainty and time. Consider

- on the one hand, a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators,
- on the other hand, a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators.

NTU, or nesting time, then uncertainty, then time, etc. We define a dynamic uncertainty criterion by the following backward induction:

$$\varrho_{T,T}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] \;, \tag{56a}$$

$$\varrho_{t,T}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \langle \mathbb{G}_t \rangle \Big[\Phi_t\Big\{ A_t, \varrho_{t+1,T}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big) \Big\} \Big] \;, \;\; \forall t \in [\![0, T-1]\!] \;. \tag{56b}$$

By the Definition 18 of $\langle \mathbb{G}_t \rangle$, we have, by construction, produced a dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$ (see Definition 4). Indeed, recalling that $A_s : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}})$, for $s \in [\![0, T]\!]$, we write

$$
\begin{aligned}
\overbrace{\varrho_{T,T}\big(A_T\big)}^{\mathcal{F}\left(\mathbb{W}_{[0:T-1]};\bar{\mathbb{R}}\right)} &= \langle \mathbb{G}_T \rangle \Big[\overbrace{A_T}^{\mathcal{F}\left(\mathbb{W}_{[0:T]};\bar{\mathbb{R}}\right)} \Big] \;, \\
\underbrace{\varrho_{t,T}\Big(\big\{A_s\big\}_{s=t}^{T}\Big)}_{\mathcal{F}\left(\mathbb{W}_{[0:t-1]};\bar{\mathbb{R}}\right)} &= \langle \mathbb{G}_t \rangle \left[\Phi_t \Big\{ \underbrace{A_t}_{\mathcal{F}\left(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}\right)} , \underbrace{\varrho_{t+1,T}\Big(\overbrace{\big\{A_s\big\}_{s=t+1}^{T}}^{\left[\mathcal{F}\left(\mathbb{W}_{[0:s]};\bar{\mathbb{R}}\right)\right]_{s=t+1}^{T}} \Big)}_{\mathcal{F}\left(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}\right)} \Big\} \right] \;, \\
& \qquad \forall t \in [\![0, T-1]\!] \;.
\end{aligned}
$$

NUT, or nesting uncertainty, then time, then uncertainty, etc. We define a dynamic uncertainty criterion by the following backward induction:

$$
\varrho_{T,T}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] \;, \tag{57a}
$$

$$
\varrho_{t,T}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \Phi_t \bigg\{ \langle \mathbb{G}_t \rangle \big[A_t\big], \langle \mathbb{G}_t \rangle \Big[\varrho_{t+1,T}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\Big] \bigg\} \;, \quad \forall t \in [\![0, T-1]\!] \;. \tag{57b}
$$

Some examples in §1.2 belong to this nested class, made of NTU and NUT.

3.2. Time-Consistency for Nested Dynamic Uncertainty Criteria

Consider

- on the one hand, a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators,
- on the other hand, a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators.

With these ingredients, we present two ways to craft a nested dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 4. For each of them, we establish time-consistency.

3.2.1. NTU Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{NTU}})(x)\big\}_{t=0}^{T}$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as the nesting

$$
\begin{aligned}
(\mathfrak{P}_t^{\mathrm{NTU}})(x) \qquad & \min_{\pi \in \Pi_t^{\mathrm{ad}}} \ \mathbb{G}_t\Bigg[\Phi_t\Bigg\{J_t\big(x_t, u_t, w_t\big), \\
& \qquad \mathbb{G}_{t+1}\Bigg[\Phi_{t+1}\Bigg\{J_{t+1}\big(x_{t+1}, u_{t+1}, w_{t+1}\big), \cdots \qquad (58a) \\
& \qquad\qquad \mathbb{G}_{T-1}\Big[\Phi_{T-1}\Big\{J_{T-1}\big(x_{T-1}, u_{T-1}, w_{T-1}\big), \\
& \qquad\qquad\qquad \mathbb{G}_T\big[J_T\big(x_T, w_T\big)\big]\Big\}\Big] \cdots \Bigg\}\Bigg]\Bigg\}\Bigg] , \\
s.t. \quad & x_t = x \ , \qquad (58b) \\
& x_{s+1} = f_s\big(x_s, u_s, w_s\big) \ , \qquad (58c) \\
& u_s = \pi_s(x_s) \ , \qquad (58d) \\
& u_s \in U_s(x_s) \ , \qquad (58e)
\end{aligned}
$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

Definition 21. We construct inductively a *NTU-dynamic uncertainty criterion* $\big\{\varrho_{t,T}^{\mathrm{NTU}}\big\}_{t=0}^{T}$ by, for any adapted uncertainty process $\big\{A_s\big\}_{s=0}^{T}$,

$$
\varrho_T^{\mathrm{NTU}}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] \ , \tag{59a}
$$

$$
\varrho_{t,T}^{\mathrm{NTU}}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \langle \mathbb{G}_t \rangle \Bigg[\Phi_t\Bigg\{A_t, \varrho_{t+1,T}^{\mathrm{NTU}}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\Bigg\}\Bigg] \ , \ \ \forall t \in [\![0, T-1]\!] \ . \tag{59b}
$$

We define the Markov optimization problem (58) formally by

$$
(\mathfrak{P}_t^{\mathrm{NTU}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{NTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \ \forall t \in [\![0, T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ , \tag{60}
$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27).

Definition 22. We define the *value functions* inductively by the DPE

$$
V_T^{\mathrm{NTU}}(x) = \mathbb{G}_T\Big[J_T(x, \cdot)\Big] \ , \ \ \forall x \in \mathbb{X}_T \ , \tag{61a}
$$

$$
V_t^{\mathrm{NTU}}(x) = \inf_{u \in U_t(x)} \mathbb{G}_t\Bigg[\Phi_t\Bigg\{J_t(x, u, \cdot), V_{t+1}^{\mathrm{NTU}} \circ f_t(x, u, \cdot)\Bigg\}\Bigg] \ , \tag{61b}
$$

$$
\forall t \in [\![0, T-1]\!] \ , \ \ \forall x \in \mathbb{X}_t \ .
$$

The following Proposition 8 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{NTU}})(x)$, for any time $t \in [\![0, T-1]\!]$ and any state $x \in \mathbb{X}_t$, can be solved by means of the value functions $\{V_t^{\mathrm{NTU}}\}_{t=0}^{T}$ in Definition 22.

Proposition 8. *Assume that*

- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Assume that there exists[10] *an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that*

$$\pi_t^\sharp(x) \in \operatorname*{arg\,min}_{u \in U_t(x)} \mathbb{G}_t \Bigg[\Phi_t \Bigg\{ J_t(x, u, \cdot), V_{t+1}^{NTU} \circ f_t(x, u, \cdot) \Bigg\} \Bigg] , \qquad \forall t \in [\![0, T-1]\!] \; , \;\; \forall x \in \mathbb{X}_t \; . \tag{62}$$

Then, $\pi^\sharp$ *is an optimal policy for any Problem* $(\mathfrak{P}_t^{NTU})(x)$*, for all* $t \in [\![0, T]\!]$ *and for all* $x \in \mathbb{X}_t$*, and*

$$V_t^{NTU}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{NTU}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big) , \qquad \forall t \in [\![0, T]\!] \; , \;\; \forall x \in \mathbb{X}_t \; . \tag{63}$$

Proof. In the proof, we drop the superscript in the value function V_t^{NTU}, that we simply denote by V_t. Let $\pi \in \Pi^{\mathrm{ad}}$ be a policy. For any $t \in [\![0, T]\!]$, we define $V_t^\pi(x)$ as the intertemporal cost from time t to time T when following policy π starting from state x:

$$V_t^\pi(x) = \varrho_{t,T}^{\mathrm{NTU}}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big) , \qquad \forall t \in [\![0, T]\!] \; , \;\; \forall x \in \mathbb{X}_t \; . \tag{64}$$

This expression is well defined because $J_{t,s}^{x,\pi} : \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}}$, for $s \in [\![t, T]\!]$ by (28).

First, we show that the functions $\{V_t^\pi\}_{t=0}^{T}$ satisfy a backward equation "à la Bellman":

$$V_t^\pi(x) = \mathbb{G}_t \Big[\Phi_t \Big\{ J_t(x, \pi_t(x), \cdot), V_{t+1}^\pi \circ f_t(x, \pi_t(x), \cdot) \Big\} \Big] \; , \;\; \forall t \in [\![0, T-1]\!] \; , \;\; \forall x \in \mathbb{X}_t \; . \tag{65}$$

Indeed, we have,

$$\begin{aligned}
V_T^\pi(x) &= \varrho_{T,T}^{\mathrm{NTU}}\Big(J_{T,T}^{x,\pi} \Big) && \text{by the definition (64) of } V_T^\pi(x), \\
&= \varrho_{T,T}^{\mathrm{NTU}}\Big(J_T(x, \cdot) \Big) && \text{by (27) that defines } J_{T,T}^{x,\pi}, \\
&= \langle \mathbb{G}_T \rangle \big[J_T(x, \cdot) \big] && \text{by the definition (59a) of } \varrho_T^{\mathrm{NTU}}, \\
&= \mathbb{G}_T \big[J_T(x, \cdot) \big] && \text{by Definition 18 of } \langle \mathbb{G}_T \rangle.
\end{aligned}$$

[10] It may be difficult to prove the existence of a measurable selection among the solutions of (62). Since it is not our intent to consider such issues, we make the assumption that an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ exists, where the definition of the set Π^{ad} is supposed to include all proper measurability conditions.

We also have, for $t \in [\![0, T-1]\!]$,

$$
\begin{aligned}
V_t^{\pi}(x) &= \varrho_{t,T}^{\mathrm{NTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \\
&\quad \text{by the definition (64) of } V_t^{\pi}(x), \\
&= \langle \mathbb{G}_t \rangle \bigg[\Phi_t\Big\{J_{t,t}^{x,\pi}, \varrho_{t+1,T}^{\mathrm{NTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t+1}^{T}\Big)\Big\}\bigg] \\
&\quad \text{by the definition (59b) of } \varrho_{t+1,T}^{\mathrm{NTU}}, \\
&= \langle \mathbb{G}_t \rangle \bigg[\Phi_t\Big\{J_{t,t}^{x,\pi}, \varrho_{t+1,T}^{\mathrm{NTU}}\Big(\big\{J_{t+1,s}^{f_t(x,\pi_t(x),\cdot),\pi}\big\}_{s=t+1}^{T}\Big)\Big\}\bigg] \\
&\quad \text{by the flow property (29),} \\
&= \langle \mathbb{G}_t \rangle \bigg[\Phi_t\Big\{J_{t,t}^{x,\pi}, V_{t+1}^{\pi} \circ f_t(x, \pi_t(x), \cdot)\Big\}\bigg] \\
&\quad \text{by the definition (64) of } V_t^{\pi}(x), \\
&= \langle \mathbb{G}_t \rangle \bigg[\Phi_t\Big\{J_t(x, \pi_t(x), \cdot), V_{t+1}^{\pi} \circ f_t(x, \pi_t(x), \cdot)\Big\}\bigg] \\
&\quad \text{by the flow property (29),} \\
&= \mathbb{G}_t \Big[\Phi_t\Big\{J_t(x, \pi_t(x), \cdot), V_{t+1}^{\pi} \circ f_t(x, \pi_t(x), \cdot)\Big\}\Big] \\
&\quad \text{by Definition 18 of } \langle \mathbb{G}_t \rangle.
\end{aligned}
$$

Second, we show that $V_t(x)$, as defined in (61) is lower than the value of the optimization problem $\mathfrak{P}_t^{\mathrm{NTU}}(x)$ in (58). For this purpose, we denote by (H_t) the following assertion

$$
(H_t): \qquad \forall x \in \mathbb{X}_t \ , \ \forall \pi \in \Pi^{\mathrm{ad}}, \qquad V_t(x) \leq V_t^{\pi}(x) \ .
$$

By definition of $V_T^{\pi}(x)$ in (64) and of $V_T(x)$ in (61a), assertion (H_T) is true.

Now, assume that (H_{t+1}) holds true. Let x be an element of $\mathbb{X}_t$. Then, by definition of $V_t(x)$ in (61b), we obtain

$$
V_t(x) \leq \inf_{\pi \in \Pi^{\mathrm{ad}}} \mathbb{G}_t \bigg[\Phi_t\Big\{J_t\big(x, \pi_t(x), \cdot\big), V_{t+1} \circ f_t\big(x, \pi_t(x), \cdot\big)\Big\}\bigg] \ , \tag{66}
$$

since, for all $\pi \in \Pi^{\mathrm{ad}}$ we have $\pi_t(x) \in U_t(x)$. By (H_{t+1}) we have, for any $\pi \in \Pi^{\mathrm{ad}}$,

$$
V_{t+1} \circ f_t\big(x, \pi_t(x), \cdot\big) \leq V_{t+1}^{\pi} \circ f_t\big(x, \pi_t(x), \cdot\big) \ .
$$

From monotonicity of Φ_t and monotonicity of $\mathbb{G}_t$, we deduce:

$$
\begin{aligned}
&\mathbb{G}_t \bigg[\Phi_t\Big\{J_t\big(x, \pi_t(x), \cdot\big), V_{t+1} \circ f_t\big(x, \pi_t(x), \cdot\big)\Big\}\bigg] \\
&\quad \leq \quad \mathbb{G}_t \bigg[\Phi_t\Big\{J_t\big(x, \pi_t(x), \cdot\big), V_{t+1}^{\pi} \circ f_t\big(x, \pi_t(x), \cdot\big)\Big\}\bigg] \ .
\end{aligned} \tag{67}
$$

We obtain:

$$
\begin{aligned}
V_t(x) &\leq \inf_{\pi \in \Pi^{\mathrm{ad}}} \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t(x),\cdot\big), V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big\}\Big] && \text{by (66),}\\
&\leq \inf_{\pi \in \Pi^{\mathrm{ad}}} \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t(x),\cdot\big), V_t^{\pi}\circ f_{t+1}\big(x,\pi_t(x),\cdot\big)\Big\}\Big] && \text{by (67),}\\
&= \inf_{\pi \in \Pi^{\mathrm{ad}}} V_t^{\pi}(x) \text{ by the definition (64) of } V_t^{\pi}(x).
\end{aligned}
$$

Hence, assertion (H_t) holds true.

Third, we show that the lower bound $V_t(x)$ for the value of the optimization problem $\mathfrak{P}_t^{\mathrm{NTU}}(x)$ is achieved for the policy $\pi^\sharp$ in (62). For this purpose, we consider the following assertion

$$
(H'_t): \qquad \forall x \in \mathbb{X}_t, \qquad V_t^{\pi^\sharp}(x) = V_t(x) .
$$

By definition of $V_T^{\pi^\sharp}(x)$ in (64) and of $V_T(x)$ in (61a), (H'_T) holds true. For $t \in [\![0, T-1]\!]$, assume that (H'_{t+1}) holds true. Let x be in $\mathbb{X}_t$. We have

$$
\begin{aligned}
V_t(x) &= \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t^\sharp(x),\cdot\big), V_{t+1}\circ f_t(x,\pi_t^\sharp(x),\cdot)\Big\}\Big] && \text{by definition of } \pi^\sharp \text{ in (62),}\\
&= \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t^\sharp(x),\cdot\big), V_{t+1}^{\pi^\sharp}\circ f_t(x,\pi_t^\sharp(x),\cdot)\Big\}\Big] && \text{by } (H'_{t+1})\\
&= V_t^{\pi^\sharp}(x) && \text{by (64).}
\end{aligned}
$$

Hence (H'_t) holds true, and the proof is complete by induction.

The following Theorem 9 is our main result on time-consistency in the NTU case.

Theorem 9. *Assume that*

- *for all $t \in [\![0, T-1]\!]$, Φ_t is non-decreasing,*
- *for all $t \in [\![0, T]\!]$, $\mathbb{G}_t$ is non-decreasing.*

Then

1. *the NTU-dynamic uncertainty criterion $\big\{\varrho_{t,T}^{NTU}\big\}_{t=0}^{T}$ defined by (59) is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}_t^{NTU})(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in (58) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (62) holds true.*

Proof. In the proof, we drop the superscripts in V_t^{NTU}, $(\mathfrak{P}_t^{\mathrm{NTU}})(x)$ and $\varrho_{t,T}^{\mathrm{NTU}}$.

The second assertion is a straightforward consequence of the property that $\pi^\sharp$ is an optimal policy[11] for *all* Problems $(\mathfrak{P}_t)(x)$. Hence, the Markov optimization problem (58) is time-consistent.

We now prove the first assertion.

Let $\underline{t} < \overline{t}$ be both in $[\![0, T]\!]$. Consider two adapted uncertainty processes $\{\underline{A}_s\}_0^T$ and $\{\overline{A}_s\}_0^T$, where $\underline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$ and $\overline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$, satisfying (39a) and (39b), that is,

$$\begin{aligned} \underline{A}_s &= \overline{A}_s \; , && \forall s \in [\![\underline{t}, \overline{t}]\!] \; , && (68a) \\ \varrho_{\overline{t},T}\big(\{\underline{A}_s\}_{\overline{t}}^T\big) &\leq \varrho_{\overline{t},T}\big(\{\overline{A}_s\}_{\overline{t}}^T\big) \; , && && (68b) \end{aligned}$$

We show by backward induction that, for all $t \in [\![\underline{t}, \overline{t}]\!]$, the following statement (H_t) holds true:

$$(H_t) \qquad \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \leq \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) \; . \tag{69}$$

First, we observe that $(H_{\overline{t}})$ holds true by assumption (68b). Second, let us assume that, for $t > \underline{t}$, the assertion (H_t) holds true. Then, by (H_t), and as $\underline{A}_{t-1} = \overline{A}_{t-1}$ by (68a), monotonicity[12] of Φ_{t-1} yields

$$\Phi_{t-1}\Big\{\underline{A}_{t-1}, \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big)\Big\} \leq \Phi_{t-1}\Big\{\overline{A}_{t-1}, \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big)\Big\} \; .$$

Monotonicity of $\mathbb{G}_{t-1}$ then gives

$$\langle \mathbb{G}_{t-1} \rangle \Big[\Phi_{t-1}\Big\{\underline{A}_{t-1}, \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big)\Big\}\Big] \leq \langle \mathbb{G}_{t-1} \rangle \Big[\Phi_{t-1}\Big\{\overline{A}_{t-1}, \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big)\Big\}\Big] \; .$$

By definition of $\varrho_{t-1,T}$ in (59), we obtain (H_{t-1}). This ends the proof by induction.

Remark 10. *As indicated in Remark 5, if we choose the inequality*

$$\forall s \in [\![\underline{t}, \overline{t}]\!], \qquad \underline{A}_s \leq \overline{A}_s \; , \tag{70}$$

as assumption to define a time-consistent dynamic uncertainty criterion (rather than the equality (43a)*), we have to make, in Theorem 9, the assumption "for all $t \in [\![0, T-1]\!]$,"*

- *"the two-variables function $(c_t, c_{t+1}) \mapsto \Phi_t(c_t, c_{t+1})$ is non-decreasing",*
- *instead of "for all c_t, the single variable function $c_{t+1} \mapsto \Phi_t(c_t, c_{t+1})$ is non-decreasing".*

[11] In all rigor, we should say that, for all $t \in [\![0, T-1]\!]$, the *tail* policy $\{\pi_s^\sharp\}_{s=t}^{T-1}$ is an optimal policy for Problem $(\mathfrak{P}_t)(x)$, for any $x \in \mathbb{X}_t$.

[12] Recall that, by Definition 13, Φ_{t-1} is non-decreasing in its second argument. Remark 10 below will enlighten this comment.

3.2.2. NUT Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{NUT}})(x)\big\}_{t=0}^{T}$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as the nesting

$$
\begin{aligned}
(\mathfrak{P}_t^{\mathrm{NUT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \quad & \Phi_t\bigg\{\mathbb{G}_t\big[J_t\big(x_t,u_t,w_t\big)\big], \mathbb{G}_t\bigg[\\
& \quad \Phi_{t+1}\Big\{\mathbb{G}_{t+1}\Big[J_{t+1}\big(x_{t+1},u_{t+1},w_{t+1}\big)\Big], \cdots \qquad (71a) \\
& \qquad \Phi_{T-1}\Big\{\mathbb{G}_{T-1}\Big[J_{T-1}\big(x_{T-1},u_{T-1},w_{T-1}\big)\Big], \\
& \qquad\quad \mathbb{G}_T\big[J_T\big(x_T,w_T\big)\big]\Big\}\cdots\Big\}\bigg]\bigg\} , \\
s.t. \quad & x_t = x \ , \qquad (71b) \\
& x_{s+1} = f_s\big(x_s,u_s,w_s\big) \ , \qquad (71c) \\
& u_s = \pi_s(x_s) \ , \qquad (71d) \\
& u_s \in U_s(x_s) \ , \qquad (71e)
\end{aligned}
$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

Definition 23. We construct inductively a *NUT-dynamic uncertainty criterion* $\big\{\varrho_{t,T}^{\mathrm{NUT}}\big\}_{t=0}^{T}$ by, for any adapted uncertainty process $\big\{A_s\big\}_{s=0}^{T}$,

$$
\varrho_T^{\mathrm{NUT}}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] \ , \tag{72a}
$$

$$
\varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \Phi_t\bigg\{ \langle \mathbb{G}_t \rangle \big[A_t\big], \langle \mathbb{G}_t \rangle \Big[\varrho_{t+1,T}^{\mathrm{NUT}}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\Big]\bigg\} , \quad \forall t \in [\![0, T-1]\!] \ . \tag{72b}
$$

We define the Markov optimization problem (71) formally by

$$
(\mathfrak{P}_t^{\mathrm{NUT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \forall t \in [\![0, T]\!] \ , \ \forall x \in \mathbb{X}_t \ , \tag{73}
$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27).

Definition 24. We define the *value functions* inductively by the DPE

$$
V_T^{\mathrm{NUT}}(x) = \mathbb{G}_T\Big[J_T(x,\cdot)\Big] \ , \ \forall x \in \mathbb{X}_T \ , \tag{74a}
$$

$$
V_t^{\mathrm{NUT}}(x) = \inf_{u \in U_t(x)} \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,u,\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^{\mathrm{NUT}} \circ f_t(x,u,\cdot)\Big]\Big\} , \quad \forall t \in [\![0, T-1]\!] \ , \ \forall x \in \mathbb{X}_t \ . \tag{74b}
$$

The following Proposition 11 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{NUT}})(x)$, for any time $t \in [\![0, T-1]\!]$ and any state $x \in \mathbb{X}_t$, can be solved by means of the value functions $\{V_t^{\mathrm{NUT}}\}_{t=0}^T$ in Definition 24.

Proposition 11. *Assume that*

- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Assume that there exists[13] *an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that*

$$\pi_t^\sharp(x) \in \mathop{\arg\min}_{u \in U_t(x)} \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,u,\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^{NUT} \circ f_t(x,u,\cdot)\Big]\Big\} \; , \quad \forall t \in [\![0, T-1]\!] \; , \; \forall x \in \mathbb{X}_t \; . \tag{75}$$

Then, $\pi^\sharp$ *is an optimal policy for any Problem* $(\mathfrak{P}_t^{NUT})(x)$, *for all* $t \in [\![0, T]\!]$ *and for all* $x \in \mathbb{X}_t$, *and*

$$V_t^{NUT}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{NUT}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^T\Big) \; , \; \forall t \in [\![0, T]\!] \; , \; \forall x \in \mathbb{X}_t \; . \tag{76}$$

Proof. In the proof, we drop the superscript in the value function V_t^{NUT}, that we simply denote by V_t. Let $\pi \in \Pi^{\mathrm{ad}}$ be a policy. For any $t \in [\![0, T]\!]$, we define $V_t^\pi(x)$ as the intertemporal cost from time t to time T when following policy π starting from state x:

$$V_t^\pi(x) = \varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^T\Big) \; , \; \forall t \in [\![0, T]\!] \; , \; \forall x \in \mathbb{X}_t \; . \tag{77}$$

This expression is well defined because $J_{t,s}^{x,\pi} : \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}}$, for $s \in [\![t, T]\!]$ by (28).

First, we show that the functions $\{V_t^\pi\}_{t=0}^T$ satisfy a backward equation "à la Bellman":

$$V_t^\pi(x) = \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,\pi_t(x),\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^\pi \circ f_t(x,\pi_t(x),\cdot)\Big]\Big\} \; , \; \forall t \in [\![0, T-1]\!] \; , \; \forall x \in \mathbb{X}_t \; . \tag{78}$$

Indeed, we have,

$$\begin{aligned}
V_T^\pi(x) &= \varrho_{T,T}^{\mathrm{NUT}}\Big(J_{T,T}^{x,\pi}\Big) && \text{by the definition (77) of } V_T^\pi(x), \\
&= \varrho_{T,T}^{\mathrm{NUT}}\Big(J_T(x,\cdot)\Big) && \text{by (27) that defines } J_{T,T}^{x,\pi}, \\
&= \langle \mathbb{G}_T \rangle \big[J_T(x,\cdot)\big] && \text{by the definition (72a) of } \varrho_T^{\mathrm{NTU}}, \\
&= \mathbb{G}_T\big[J_T(x,\cdot)\big] && \text{by Definition 18 of } \langle \mathbb{G}_T \rangle.
\end{aligned}$$

[13]See Footnote 10.

We also have, for $t \in [\![0, T-1]\!]$,

$$
\begin{aligned}
V_t^{\pi}(x) &= \varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \\
&\qquad \text{by the definition (77) of } V_t^{\pi}(x), \\
&= \Phi_t\Big\{\langle\mathbb{G}_t\rangle\Big[J_{t,t}^{x,\pi}\Big], \langle\mathbb{G}_t\rangle\Big[\varrho_{t+1,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t+1}^{T}\Big)\Big]\Big\} \\
&\qquad \text{by the definition (72b) of } \varrho_{t+1,T}^{\mathrm{NUT}}, \\
&= \Phi_t\Big\{\langle\mathbb{G}_t\rangle\Big[J_{t,t}^{x,\pi}\Big], \langle\mathbb{G}_t\rangle\Big[\varrho_{t+1,T}^{\mathrm{NUT}}\Big(\big\{J_{t+1,s}^{f_t(x,\pi_t(x),\cdot),\pi}\big\}_{s=t+1}^{T}\Big)\Big]\Big\} \\
&\qquad \text{by the flow property (29)} \\
&= \Phi_t\Big\{\langle\mathbb{G}_t\rangle\Big[J_{t,t}^{x,\pi}\Big], \langle\mathbb{G}_t\rangle\Big[V_{t+1}^{\pi}\circ f_t(x,\pi_t(x),\cdot)\Big]\Big\} \\
&\qquad \text{by the definition (77) of } V_t^{\pi}(x), \\
&= \Phi_t\Big\{\langle\mathbb{G}_t\rangle\Big[J_t(x,\pi_t(x),\cdot)\Big], \langle\mathbb{G}_t\rangle\Big[V_{t+1}^{\pi}\circ f_t(x,\pi_t(x),\cdot)\Big]\Big\} \\
&\qquad \text{by the flow property (29)} \\
&= \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,\pi_t(x),\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^{\pi}\circ f_t(x,\pi_t(x),\cdot)\Big]\Big\} \\
&\qquad \text{by Definition 18 of } \langle\mathbb{G}_t\rangle.
\end{aligned}
$$

Second, we show that $V_t(x)$, as defined in (74) is lower than the value of the optimization problem $\mathfrak{P}_t^{\mathrm{NUT}}(x)$ in (71). For this purpose, we denote by (H_t) the following assertion

$$(H_t): \qquad \forall x \in \mathbb{X}_t\ ,\ \forall \pi \in \Pi^{\mathrm{ad}}, \qquad V_t(x) \leq V_t^{\pi}(x)\ .$$

By definition of $V_T^{\pi}(x)$ in (77) and of $V_T(x)$ in (74a), assertion (H_T) is true.

Now, assume that (H_{t+1}) holds true. Let x be an element of $\mathbb{X}_t$. Then, by definition of $V_t(x)$ in (74b), we obtain

$$V_t(x) \leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big], \mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\}\ , \tag{79}$$

since, for all $\pi \in \Pi^{\mathrm{ad}}$ we have $\pi_t(x) \in U_t(x)$. By (H_{t+1}) we have, for any $\pi \in \Pi^{\mathrm{ad}}$,

$$V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big) \leq V_{t+1}^{\pi}\circ f_t\big(x,\pi_t(x),\cdot\big)\ .$$

From monotonicity of Φ_t and monotonicity of $\mathbb{G}_t$, we deduce:

$$
\begin{aligned}
&\Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big], \mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \\
\leq\ &\Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big], \mathbb{G}_t\Big[V_{t+1}^{\pi}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\}\ .
\end{aligned} \tag{80}
$$

We obtain:

$$\begin{aligned}
V_t(x) &\leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \text{ by (79)},\\
&\leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V^{\pi}_{t+1}\circ f_{t+1}\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \text{ by (80)},\\
&= \inf_{\pi\in\Pi^{\mathrm{ad}}} V^{\pi}_t(x) \text{ by the definition (77) of } V^{\pi}_t(x).
\end{aligned}$$

Hence (H_t) holds true.

Third, we show that the lower bound $V_t(x)$ for the value of the optimization problem $\mathfrak{P}^{\mathrm{NUT}}_t(x)$ is achieved for the policy $\pi^\sharp$ in (75). For this purpose, we consider the following assertion

$$(H'_t): \qquad \forall x\in\mathbb{X}_t, \qquad V^{\pi^\sharp}_t(x) = V_t(x)\ .$$

By definition of $V^{\pi^\sharp}_T(x)$ in (77) and of $V_T(x)$ in (74a), (H'_T) holds true. For $t\in[\![0,T-1]\!]$, assume that (H'_{t+1}) holds true. Let x be in $\mathbb{X}_t$. We have

$$\begin{aligned}
V_t(x) &= \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi^\sharp_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} && \text{by definition of } \pi^\sharp \text{ in (75)},\\
&= \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi^\sharp_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V^{\pi^\sharp}_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} && \text{by } (H'_{t+1})\\
&= V^{\pi^\sharp}_t(x) && \text{by (77).}
\end{aligned}$$

Hence (H'_t) holds true, and the proof is complete by induction.

The following Theorem 12 is our main result on time-consistency in the NUT case.

Theorem 12. *Assume that*

- *for all* $t\in[\![0,T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t\in[\![0,T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Then

1. *the NUT-dynamic uncertainty criterion* $\big\{\varrho^{NUT}_{t,T}\big\}_{t=0}^{T}$ *defined by* (72) *is time-consistent;*
2. *the Markov optimization problem* $\Big\{\big\{(\mathfrak{P}^{NUT}_t)(x)\big\}_{x\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ *defined in* (71) *is time-consistent, as soon as there exists an admissible policy* $\pi^\sharp\in\Pi^{\mathrm{ad}}$ *such that* (75) *holds true.*

Proof. In the proof, we drop the superscripts in V^{NUT}_t, $(\mathfrak{P}^{\mathrm{NUT}}_t)(x)$ and $\varrho^{\mathrm{NUT}}_{t,T}$.

The second assertion is a straightforward consequence of the property that $\pi^\sharp$ is an optimal policy[14] for *all* Problems $(\mathfrak{P}_t)(x)$. Hence, the Markov optimization problem (71) is time-consistent.

We now prove the first assertion. We suppose given a policy $\pi \in \Pi$, and a sequence $\{x_s\}_0^T$ of states, where $x_s \in \mathbb{X}_s$.

Let $\underline{t} < \overline{t}$ be both in $[\![0, T]\!]$. Consider two adapted uncertainty processes $\{\underline{A}_s\}_0^T$ and $\{\overline{A}_s\}_0^T$, where $\underline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$ and $\overline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$, satisfying (39a) and (39b), that is,

$$\begin{aligned}
\underline{A}_s &= \overline{A}_s \;, & \forall s \in [\![\underline{t}, \overline{t}]\!] \;, && \text{(81a)}\\
\varrho_{\overline{t},T}\big(\{\underline{A}_s\}_{\overline{t}}^T\big) &\leq \varrho_{\overline{t},T}\big(\{\overline{A}_s\}_{\overline{t}}^T\big) \;, & && \text{(81b)}
\end{aligned}$$

We show by backward induction that, for all $t \in [\![\underline{t}, \overline{t}]\!]$, the following statement (H_t) holds true:

$$(H_t) \qquad \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \quad \leq \quad \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) \;. \tag{82}$$

First, we observe that $(H_{\overline{t}})$ holds true by assumption (81b). Second, let us assume that, for $t > \underline{t}$, the assertion (H_t) holds true. Then, by (H_t), monotonicity of $\mathbb{G}_{t-1}$ gives

$$\langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \Big] \quad \leq \quad \langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) \Big] \;.$$

As $\underline{A}_{t-1} = \overline{A}_{t-1}$ by (81a), monotonicity[15] of Φ_{t-1} yields

$$\Phi_{t-1}\Big\{ \underline{A}_{t-1}, \langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \Big] \Big\} \quad \leq \quad \Phi_{t-1}\Big\{ \overline{A}_{t-1}, \langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) \Big] \Big\} \;.$$

By definition of $\varrho_{t-1,T}$ in (72), we obtain (H_{t-1}). This ends the proof by induction.

3.3. Commutation of Aggregators

We introduce two notions of commutation between time and uncertainty aggregators.

3.3.1. TU-Commutation of Aggregators

The following notion of TU-commutation between time and uncertainty aggregators stands as one of the key ingredients for a DPE.

Definition 25. Let $t \in [\![0, T]\!]$ and $s \in [\![t+1, T]\!]$. A $[t\!:\!s]$-multiple-step uncertainty-aggregator $\mathbb{G}^{[t:s]}$ is said to *TU-commute* with a one-step time-aggregator Φ if

$$\mathbb{G}^{[t:s]}\Big[w_{[t:s]} \mapsto \Phi\big\{ c, D_t\big(w_{[t:s]}\big) \big\} \Big] = \Phi\Big\{ c, \mathbb{G}^{[t:s]}\Big[w_{[t:s]} \mapsto D_t\big(w_{[t:s]}\big) \Big] \Big\} \;, \tag{83}$$

for any function $D_t \in \mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}})$ and any extended scalar $c \in \bar{\mathbb{R}}$.

[14]See Footnote 11.
[15]See Footnote 12.

In particular, a one-step time-aggregator Φ TU-commutes with a one-step uncertainty-aggregator $\mathbb{G}^{[t:t]}$ if

$$\mathbb{G}^{[t:t]}\Big[\Phi\big\{c, C_t\big\}\Big] = \Phi\Big\{c, \mathbb{G}^{[t:t]}\big[C_t\big]\Big\} , \tag{84}$$

for any function[16] $C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$ and any extended scalar $c \in \bar{\mathbb{R}}$.

Example 13. *If $(\mathbb{W}_t, \mathcal{F}_t, \mathbb{P}_t)$ is a probability space and if*

$$\Phi\big\{c, c_t\big\} = \alpha(c) + \beta(c) c_t , \tag{85}$$

where $\alpha : \bar{\mathbb{R}} \to \mathbb{R}$ and $\beta : \bar{\mathbb{R}} \to \mathbb{R}_+$, then the extended[17] expectation $\mathbb{G}^{[t:t]} = \mathbb{E}_{\mathbb{P}_t}$ TU-commutes with Φ.

Proposition 14. *Consider a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators and a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators. Suppose that, for any $0 \le t < s \le T$, $\mathbb{G}_s$ TU-commutes with Φ_t.*

Then, $\Big\langle \boxdot_{s=t}^{T} \mathbb{G}_s \Big\rangle$ TU-commutes with Φ_r, for any $0 \le r < t \le T$, that is,

$$\Big\langle \boxdot_{s=t}^{T} \mathbb{G}_s \Big\rangle \Big[\Phi_r\big\{c_r, A\big\}\Big] = \Phi_r\Big\{c, \Big\langle \boxdot_{s=t}^{T} \mathbb{G}_s \Big\rangle \big[A\big]\Big\} , \qquad \forall\, 0 \le r < t \le T , \tag{86}$$

for any extended scalar $c \in \bar{\mathbb{R}}$ and any function $A \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

Proof. We prove by induction that

$$\Big(\boxdot_{s=t}^{T} \mathbb{G}_s \Big) \Big[\Phi_r\big\{c, D_t\big\}\Big] = \Phi_r\Big\{c, \Big(\boxdot_{s=t}^{T} \mathbb{G}_s \Big) \big[D_t\big]\Big\} , \qquad \forall\, 0 \le r < t \le T , \tag{87}$$

for any extended scalar $c \in \bar{\mathbb{R}}$ and any function $D_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big)$. For $t \in [\![1, T]\!]$, let (H_t) be the following assertion

$$\begin{aligned} (H_t) : \quad & \forall r \in [\![0, t-1]\!] ,\ \forall c \in \bar{\mathbb{R}} ,\ \forall D_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big) , \\ & \Big(\boxdot_{s=t}^{T} \mathbb{G}_s \Big) \Big[\Phi_r\big\{c, D_t\big\}\Big] = \Phi_r\Big\{c, \Big(\boxdot_{s=t}^{T} \mathbb{G}_s \Big) \big[D_t\big]\Big\} . \end{aligned} \tag{88}$$

The assertion (H_T) is

$$\begin{aligned} (H_T) : \quad & \forall r \in [\![0, T-1]\!] ,\ \forall c \in \bar{\mathbb{R}} ,\ \forall D_T \in \mathcal{F}\big(\mathbb{W}_T; \bar{\mathbb{R}}\big), \\ & \mathbb{G}_T\Big[\Phi_r\big\{c, D_T\big\}\Big] = \Phi_r\big\{c, \mathbb{G}_T\big[D_T\big]\big\} . \end{aligned}$$

---

[16]We will consistently use the symbol C_t to denote a function in $\mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$, that is, $C_t : \mathbb{W}_t \to \bar{\mathbb{R}}$.

[17]We set $\beta \ge 0$, so that, when $C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$ is not integrable with respect to $\mathbb{P}_t$, the equality (83) still holds true.

Thus, the assertion (H_T) is true, since it coincides the property that, for any $0 \leq r < T$, $\mathbb{G}_T$ TU-commutes with Φ_r (apply (83) where $t = T$, $\Phi = \Phi_r$).

Now, suppose that (H_{t+1}) holds true. Let $r < t$, $c \in \bar{\mathbb{R}}$ and $D_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big)$. We have

$$
\begin{aligned}
&\Big(\mathop{\boxdot}_{s=t}^{T} \mathbb{G}_s\Big)\Big[\Phi_r\big\{c, D_t\big\}\Big] ,\\
&\quad = \quad \mathbb{G}_t\Bigg[w_t \mapsto \Big(\mathop{\boxdot}_{t+1}^{T} \mathbb{G}_s\Big)\Big[w_{[t+1:T]} \mapsto \Phi_r\Big\{c, D_t\big(w_t, w_{[t+1:T]}\big)\Big\}\Big]\Bigg] ,\\
&\qquad \text{by the definition (53) of composition,}\\
&\quad = \quad \mathbb{G}_t\Bigg[w_t \mapsto \Phi_r\Big\{c, \Big(\mathop{\boxdot}_{s=t+1}^{T} \mathbb{G}_s\Big)\Big[w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big)\Big]\Big\}\Bigg]\\
&\qquad \text{by } (H_{t+1}) \text{ since } r < t < t+1,\\
&\qquad \text{and where, for all } w_t,\ D_{t+1} : w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big) \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big) ,\\
&\quad = \quad \Phi_r\Bigg\{c, \mathbb{G}_t\Bigg[w_t \mapsto \Big(\mathop{\boxdot}_{s=t+1}^{T} \mathbb{G}_s\Big)\Big[w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big)\Big]\Bigg]\Bigg\} ,\\
&\qquad \text{by commutation property (83) of } \mathbb{G}_t \text{ with } \Phi = \Phi_r, \text{ since } 0 \leq r < t \leq T,\\
&\qquad \text{and where } C_t : w_t \mapsto \Big(\mathop{\boxdot}_{s=t+1}^{T} \mathbb{G}_s\Big)\big[w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big)\big] \in \mathcal{F}\big(\mathbb{W}_t; \bar{\mathbb{R}}\big) ,\\
&\quad = \quad \Phi_r\Big\{c, \Big(\mathop{\boxdot}_{s=t}^{T} \mathbb{G}_s\Big)\big[D_t\big]\Big\} \text{ by the definition (53) of composition.}
\end{aligned}
$$

This ends the induction, hence the proof of (87). Then, (86) easily follows by the extensions of Definitions 16 and 18.

3.3.2. UT-Commutation of Aggregators

The following notion of UT-commutation between time and uncertainty aggregators stands as one of the key ingredients for a DPE. In practice, it is much more restrictive than TU-commutation.

Definition 26. Let $t \in [\![0, T]\!]$. A multiple-step time-aggregator $\Phi : \bar{\mathbb{R}}^{k+1} \to \bar{\mathbb{R}}$ is said to *UT-commute* with a one-step uncertainty-aggregator $\mathbb{G}^{[t:t]}$ if

$$
\Big\langle \mathbb{G}^{[t:t]} \Big\rangle \Big[\Phi\Big(\big\{A_s\big\}_{s=0}^{k}\Big)\Big] = \Phi\Big(\Big\{ \Big\langle \mathbb{G}^{[t:t]} \Big\rangle \big[A_s\big] \Big\}_{s=0}^{k}\Big) , \tag{89}
$$

for any adapted uncertainty process $\big\{A_s\big\}_{s=0}^{k}$.

In particular, a one-step time-aggregator Φ UT-commutes with a one-step uncertainty-aggregator $\mathbb{G}^{[t:t]}$ if

$$
\mathbb{G}^{[t:t]}\Big[\Phi\big\{B_t, C_t\big\}\Big] = \Phi\Big\{\mathbb{G}^{[t:t]}\big[B_t\big], \mathbb{G}^{[t:t]}\big[C_t\big]\Big\} , \tag{90}
$$

for any functions B_t, C_t in $\mathcal{F}(\mathbb{W}_t;\bar{\mathbb{R}})$. Comparing (90) with (84), we observe that UT-commutation requires a property bearing on the first argument of the one-step time-aggregator Φ, whereas TU-commutation does not. In practical applications, UT-commutation is much more restrictive than TU-commutation.

Example 15. *If $(\mathbb{W}_t, \mathcal{F}_t, \mathbb{P}_t)$ is a probability space, then the extended expectation $\mathbb{G}^{[t:t]} = \mathbb{E}_{\mathbb{P}_t}$ UT-commutes with Φ, given by $\Phi\big\{c, c_t\big\} = \alpha(c)+\beta(c)c_t$ in (85), only in the case where α is linear and β is a constant. Comparing with Example 13, UT-commutation appears much more restrictive than TU-commutation.*

Proposition 16. *Consider a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators and a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators. Suppose that, for any $0 \leq t < s \leq T$, Φ_s TU-commutes with $\mathbb{G}_t$.*

Then, $\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ TU-commutes with $\mathbb{G}_r$, for any $r \in [\![0, t-1]\!]$, that is, for any $\big\{A_s\big\}_{s=t}^{T}$, where $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:T]};\bar{\mathbb{R}}\big)$,

$$\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \Big\{ \mathbb{G}_r\big[A_s\big] \Big\}_{s=t}^{T} \Big\} = \mathbb{G}_r\Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{A_s\big\}_{s=t}^{T} \Big\} \Big] \; , \; \forall 0 \leq r < t \leq T \; . \tag{91}$$

Proof. We prove by induction that

$$\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \Big\{ \mathbb{G}_r\big[C_s\big] \Big\}_{s=t}^{T} \Big\} = \mathbb{G}_r\Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{C_s\big\}_{s=t}^{T} \Big\} \Big] \; , \; \forall 0 \leq r < t \leq T \; , \tag{92}$$

for any $\big\{C_s\big\}_{s=t}^{T}$, where $C_s \in \mathcal{F}\big(\mathbb{W}_r;\bar{\mathbb{R}}\big)$.

For $t \in [\![0, T-1]\!]$, let (H_t) be the following assertion

$$\begin{aligned}(H_t): \quad & \forall r \in [\![0, t-1]\!] \; , \; \forall s \in [\![t, T]\!] \; , \; \forall C_s \in \mathcal{F}\big(\mathbb{W}_r;\bar{\mathbb{R}}\big) \; , \\ & \Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \Big\{ \mathbb{G}_r\big[C_s\big] \Big\}_{s=t}^{T} \Big\} = \mathbb{G}_r\Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{C_s\big\}_{s=t}^{T} \Big\} \Big] \; .\end{aligned} \tag{93}$$

The assertion (H_{T-1}) is

$$\begin{aligned}(H_{T-1}): \quad & \forall r \in [\![0, T-2]\!] \; , \; \forall C_T \in \mathcal{F}\big(\mathbb{W}_r;\bar{\mathbb{R}}\big) \; , \; \forall C_{T-1} \in \mathcal{F}\big(\mathbb{W}_r;\bar{\mathbb{R}}\big) \; , \\ & \langle \Phi_{T-1} \rangle \Big\{ \mathbb{G}_r\big[C_{T-1}\big], \mathbb{G}_r\big[C_T\big] \Big\} = \mathbb{G}_r\Big[\langle \Phi_{T-1} \rangle \big\{ C_{T-1}, C_T \big\} \Big] \; .\end{aligned} \tag{94}$$

Thus, the assertion (H_{T-1}) is true, since it coincides the property that, for any $0 \leq r < T$, Φ_{T-1} TU-commutes with $\mathbb{G}_r$ (apply (89) where $t = T$, $\Phi = \Phi_{T-1}$, $A_s = C_s$).

Now, suppose that (H_{t+1}) holds true. With $r < t$, and $C_s \in \mathcal{F}\big(\mathbb{W}_r;\bar{\mathbb{R}}\big)$, for all $s \in [\![t, T]\!]$, we have

$$
\begin{aligned}
\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ \mathbb{G}_r\big[C_s\big] \big\}_{s=t}^{T} \Big\} &= \Phi_t \Big\{ \mathbb{G}_r\big[C_t\big], \Big\langle \bigodot_{s=t+1}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ \mathbb{G}_r\big[C_s\big] \big\}_{s=t+1}^{T} \Big\} \Big\} \\
&\quad \text{by the definition (45) of composition,} \\
&= \Phi_t \Big\{ \mathbb{G}_r\big[C_t\big], \mathbb{G}_r\Big[\Big\langle \bigodot_{s=t+1}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t+1}^{T} \Big\} \Big] \Big\} \\
&\quad \text{by } (H_{t+1}) \text{ since } r < t < t+1 \\
&= \mathbb{G}_r\Big[\Phi_t \Big\{ C_t, \Big\langle \bigodot_{s=t+1}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t+1}^{T} \Big\} \Big\} \Big] \\
&\quad \text{by commutation property (89) of } \mathbb{G}_r \text{ with } \Phi = \Phi_t \\
&\quad \text{since } 0 \leq r < t \leq T, \\
&= \mathbb{G}_r\Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t}^{T} \Big\} \Big] \\
&\quad \text{by the definition (45) of composition.}
\end{aligned}
$$

This ends the induction, hence the proof of (92). The property that $\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ TU-commutes with $\mathbb{G}_r$, for any $r \in [\![0, t-1]\!]$, easily follows by the extensions of Definitions 16 and 18.

3.4. Time-Consistency for Non Nested Dynamic Uncertainty Criteria

Consider

- on the one hand, a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators,
- on the other hand, a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators.

With these ingredients, and with the compositions $\Big(\boxdot_{s=t}^{T} \mathbb{G}_s \Big)$ and $\Big\langle \boxdot_{s=t}^{T} \mathbb{G}_s \Big\rangle$ introduced in Definitions 20 and 18, and $\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ in Definition 16, we present two ways to craft a non-nested dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 4. For each of them, we provide a DPE under the assumption that time and uncertainty aggregators commute.

3.4.1. TU Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{TU}})(x)\big\}_{t=0}^{T}$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as

$$
\begin{aligned}
(\mathfrak{P}_t^{\mathrm{TU}})(x) \qquad & \min_{\pi \in \Pi_t^{\mathrm{ad}}} \ \mathbb{G}_t\Bigg[\mathbb{G}_{t+1}\Bigg[\cdots \mathbb{G}_T\Bigg[\\
& \qquad \Phi_t\Big\{J_t\big(x_t,u_t,w_t\big), \\
& \qquad\quad \Phi_{t+1}\Big\{J_{t+1}\big(x_{t+1},u_{t+1},w_{t+1}\big),\cdots \qquad (95a) \\
& \qquad\qquad \Phi_{T-1}\Big\{J_{T-1}\big(x_{T-1},u_{T-1},w_{T-1}\big), J_T\big(x_T,w_T\big)\Big\} \\
& \qquad\qquad\qquad \cdots\Big\}\Big\}\Bigg]\cdots\Bigg]\Bigg] , \\
s.t. \quad & x_t = x \ , \qquad (95b) \\
& x_{s+1} = f_s\big(x_s,u_s,w_s\big) \ , \qquad (95c) \\
& u_s = \pi_s(x_s) \ , \qquad (95d) \\
& u_s \in U_s(x_s) \ , \qquad (95e)
\end{aligned}
$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

We define the Markov optimization problem (95) formally by

$$
(\mathfrak{P}_t^{\mathrm{TU}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{TU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ , \qquad (96)
$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27), and where $\varrho_{t,T}^{\mathrm{TU}}$ is defined as follows.

When we compose

$$
\big[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\big]_{s=t}^{T} \xrightarrow{\left\langle \bigodot_{s=t}^{T-1} \Phi_s \right\rangle} \mathcal{F}(\mathbb{W}_{[0:T]};\bar{\mathbb{R}}) \xrightarrow{\left\langle \boxdot_{s=t}^{T} \mathbb{G}_s \right\rangle} \mathcal{F}(\mathbb{W}_{[0:t-1]};\bar{\mathbb{R}}) \ , \qquad (97)
$$

we obtain the following Definition.

Definition 27. We define the dynamic uncertainty criterion $\{\varrho_{t,T}^{\mathrm{TU}}\}_{t=0}^{T}$ by[18]

$$
\varrho_{t,T}^{\mathrm{TU}} = \Big\langle \boxdot_{s=t}^{T} \mathbb{G}_s \Big\rangle \circ \Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \ , \ \ \forall t \in [\![0, T-1]\!] \ . \qquad (98)
$$

When we plug the stream $\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}$ of costs, introduced in Definition 3, into the operator above, this two-stage process displays a natural economic interpretation in term of preferences: we mix time and uncertainty preferences, first with respect to time, then with respect to uncertainty.

[18]With the convention that $\Big(\bigodot_{r=T}^{T-1} \Phi_r\Big)$ is the identity mapping.

- We aggregate streams $\left\{J_{t,s}^{x,\pi}(w)\right\}_{s=t}^{T}$ of costs, *first with respect to time*, thanks to the function $\left(\bigodot_{s=t}^{T-1}\Phi_s\right) : \bar{\mathbb{R}}^{T+1} \to \bar{\mathbb{R}}$. However, the result $\left(\bigodot_{s=t}^{T-1}\Phi_s\right)\left(\left\{J_{t,s}^{x,\pi}(w)\right\}_{s=t}^{T}\right)$ still depends upon the scenario w.
- Then, we aggregate uncertain intertemporal costs $w \mapsto \left(\bigodot_{s=t}^{T-1}\Phi_s\right)\left(\left\{J_{t,s}^{x,\pi}(w)\right\}_{s=t}^{T}\right)$ — elements of the set $\mathcal{F}(\mathbb{W}_{[t:T]};\bar{\mathbb{R}})$ of functions — *second with respect to uncertainty*, thanks to the multiple-step uncertainty-aggregator $\boxdot_{s=t}^{T}\mathbb{G}_s : \mathcal{F}(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.

The following Theorem 17 is our main result on time-consistency in the TU case.

Theorem 17. *Assume that*

- *for any* $0 \leq s < t \leq T$, $\mathbb{G}_t$ *TU-commutes with* Φ_s,
- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Then

1. *the TU-dynamic uncertainty criterion* $\left\{\varrho_{t,T}^{TU}\right\}_{t=0}^{T}$ *defined by* (98) *is time-consistent;*
2. *the Markov optimization problem* $\left\{\left\{(\mathfrak{P}_t^{TU})(x)\right\}_{x\in\mathbb{X}_t}\right\}_{t=0}^{T}$ *defined in* (95) *is time-consistent, as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that* (62) *holds true, where the value functions are the* $\left\{V_t^{NTU}\right\}_{t=0}^{T}$ *in Definition 22.*

Proof. Since, for any $0 \leq s < t \leq T$, $\mathbb{G}_t$ TU-commutes with Φ_s, the TU-dynamic uncertainty criterion $\{\varrho_{t,T}^{\mathrm{TU}}\}_{t=0}^{T}$, given by Definition 27, coincides with $\{\varrho_{t,T}^{\mathrm{NTU}}\}_{t=0}^{T}$, given by Definition 21. Indeed, we prove that $\{\varrho_{t,T}^{\mathrm{TU}}\}_{t=0}^{T}$ satisfies the backward induction (59).

With the convention[19] that $\left(\bigodot_{r=T}^{T-1}\Phi_r\right)$ is the identity mapping, we have

[19]See Footnote 18

$\varrho_T^{\mathrm{TU}} = \langle \mathbb{G}_T \rangle$, that is, (59a). For any $\{A_s\}_t^T \in [\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})]_{s=t}^T$, we have:

$$\begin{aligned}
\varrho_t^{\mathrm{TU}}\Big(\big\{A_s\big\}_{s=t}^T\Big) = \quad & \Big\langle \mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big[\Big\langle \mathop{\odot}_{r=t}^{T-1} \Phi_r \Big\rangle \Big\{ \big\{A_s\big\}_{s=t}^T \Big\} \Big] \text{ by (98),} \\
= \quad & \mathbb{G}_t \Big[\Big\langle \mathop{\boxdot}_{r=t+1}^{s} \mathbb{G}_r \Big\rangle \Big[\Big\langle \mathop{\odot}_{r=t}^{T-1} \Phi_r \Big\rangle \Big\{ \big\{A_s\big\}_{s=t}^T \Big\} \Big] \Big] \text{ by (53),} \\
= \quad & \mathbb{G}_t \Big[\Big\langle \mathop{\boxdot}_{r=t+1}^{s} \mathbb{G}_r \Big\rangle \Phi_t \Big\{ A_t, \Big(\mathop{\odot}_{r=t+1}^{T-1} \Phi_r \Big) \big\{A_s\big\}_{s=t+1}^T \Big\} \Big] \text{ by (45),} \\
= \quad & \mathbb{G}_t \Big[\Phi_t \Big\{ A_t, \Big\langle \mathop{\boxdot}_{r=t+1}^{s} \mathbb{G}_r \Big\rangle \Big[\Big(\mathop{\odot}_{r=t+1}^{T-1} \Phi_r \Big) \big\{A_s\big\}_{s=t+1}^T \Big] \Big\} \Big] \\
& \text{by commutation property (91),} \\
= \quad & \mathbb{G}_t \Big[\Phi_t \Big(A_t, \varrho_{t+1}^{\mathrm{TU}}\Big(\big\{A_s\big\}_{s=t+1}^T \Big) \Big) \Big] \text{ by (98).}
\end{aligned}$$

660 *3.4.2. UT Dynamic Uncertainty Criterion*

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{UT}})(x)\big\}_{t=0}^T$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as

$$\begin{aligned}
(\mathfrak{P}_t^{\mathrm{UT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \quad & \Phi_t \Big\{ \mathbb{G}_t \Big[J_t\big(x_t, u_t, w_t\big) \Big], \\
& \quad \Phi_{t+1} \Big\{ \mathbb{G}_t \mathbb{G}_{t+1} \Big[J_{t+1}\big(x_{t+1}, u_{t+1}, w_{t+1}\big), \cdots \\
& \qquad \Phi_{T-1} \Big\{ \mathbb{G}_t \cdots \mathbb{G}_{T-1} \Big[J_{T-1}\big(x_{T-1}, u_{T-1}, w_{T-1}\big) \Big], \\
& \qquad\qquad \mathbb{G}_t \cdots \mathbb{G}_T \big[J_T\big(x_T, w_T\big) \big] \Big\} \Big] \cdots \Big\} \Big\} ,
\end{aligned} \tag{99a}$$

$$s.t. \quad x_t = x \; , \tag{99b}$$

$$x_{s+1} = f_s\big(x_s, u_s, w_s\big) \; , \tag{99c}$$

$$u_s = \pi_s(x_s) \; , \tag{99d}$$

$$u_s \in U_s(x_s) \; , \tag{99e}$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

We define the Markov optimization problem (99) formally by

$$(\mathfrak{P}_t^{\mathrm{UT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{UT}}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^T \Big) \; , \;\; \forall t \in [\![0, T]\!] \; , \;\; \forall x \in \mathbb{X}_t \; , \tag{100}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27), and where $\varrho_{t,T}^{\mathrm{UT}}$ is defined as follows.

We define the mapping

$$\Big\{ \Big(\mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big) \Big\}_{s=t}^{T} : \big[\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \to \bar{\mathbb{R}}^{T+1} , \tag{101}$$

for any $\big\{ D_r \big\}_{r=t}^{T} \in \big[\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T}$, componentwise by

$$\Big(\mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big) \Big[\big\{ D_s \big\}_{s=t}^{T} \Big] = \Big\{ \Big(\mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big) \big[D_s \big] \Big\}_{s=t}^{T} . \tag{102}$$

In the same way, we define the mapping (see Definition 18):

$$\Big\{ \Big\langle \mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big\}_{s=t}^{T} : \big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \to \Big(\mathcal{F}(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}) \Big)^{T+1} . \tag{103}$$

Definition 28. We define the dynamic uncertainty criterion $\{\varrho_{t,T}^{\mathrm{UT}}\}_{t=0}^{T}$ by

$$\varrho_{t,T}^{\mathrm{UT}} = \Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \circ \Big\{ \Big\langle \mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big\}_{s=t}^{T} , \ \ \forall t \in [\![0, T-1]\!] . \tag{104}$$

The expression $\varrho_{t,T}^{\mathrm{UT}}$ is the output of the composition[20]

$$\big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \xrightarrow{\Big\{ \Big\langle \mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big\}_{s=t}^{T}} \Big(\mathcal{F}(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}) \Big)^{T+1} \xrightarrow{\Big(\bigodot_{s=t}^{T-1} \Phi_s \Big)} \mathcal{F}(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}) .$$

When we plug the stream $\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T}$ of costs, introduced in Definition 3, into the operator above, this two-stage process displays a natural economic interpretation in term of preferences: we mix time and uncertainty preferences, first with respect to uncertainty, then with respect to time.

- We aggregate the stream $\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T}$ of uncertain costs, *first with respect to uncertainty*, producing

$$\Big\{ \mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \big[J_{t,s}^{x,\pi} \big] \Big\}_{s=t}^{T} = \Big\{ \mathbb{G}_t \Big[J_{t,t}^{x,\pi} \Big], \ldots, \Big(\mathop{\boxdot}\limits_{r=t}^{T} \mathbb{G}_r \Big) \big[J_{t,T}^{x,\pi} \big] \Big\} , \tag{105}$$

  thanks to the multiple-step uncertainty-aggregators $\mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r : \mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}) \to \bar{\mathbb{R}}$, for $s \in [\![t, T]\!]$. However, the resulting quantity $\Big\{ \Big(\mathop{\boxdot}\limits_{r=t}^{s} \mathbb{G}_r \Big) \big[J_{t,s}^{x,\pi} \big] \Big\}_{s=t}^{T}$ still depends upon time s.

[20]With the convention that $\mathcal{F}(\mathbb{W}_{[0:-1]}; \bar{\mathbb{R}}) = \bar{\mathbb{R}}$, we have $\varrho_0^{\mathrm{UT}} : \big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \to \bar{\mathbb{R}}$.

- Then, we aggregate the time sequence $\Big\{ \Big(\mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big) \big[J_{t,s}^{x,\pi} \big] \Big\}_{s=t}^{T}$ of costs, *second with respect to time*, thanks to $\Big(\bigodot_{r=t}^{T-1} \Phi_r \Big) : \bar{\mathbb{R}}^{T+1} \to \bar{\mathbb{R}}$.

The following Theorem 18 is our main result on time-consistency in the UT case.

Theorem 18. *Assume that*

- *for any* $0 \leq s < t \leq T$, $\mathbb{G}_t$ *UT-commutes with* Φ_s,
- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Then

1. *the UT-dynamic uncertainty criterion* $\big\{ \varrho_{t,T}^{UT} \big\}_{t=0}^{T}$ *defined by* (104) *is time-consistent;*
2. *the Markov optimization problem* $\Big\{ \big\{ (\mathfrak{P}_t^{UT})(x) \big\}_{x \in \mathbb{X}_t} \Big\}_{t=0}^{T}$ *defined in* (99) *is time-consistent, as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that* (75) *holds true, where the value functions are the* $\big\{ V_t^{NUT} \big\}_{t=0}^{T}$ *in Definition 24.*

Proof. Since, for any $0 \leq s < t \leq T$, $\mathbb{G}_t$ UT-commutes with Φ_s, the UT-dynamic uncertainty criterion $\{\varrho_{t,T}^{\mathrm{UT}}\}_{t=0}^{T}$, given by Definition 28, coincides with $\{\varrho_{t,T}^{\mathrm{NUT}}\}_{t=0}^{T}$, given by Definition 23.

Indeed, we prove that $\{\varrho_{t,T}^{\mathrm{UT}}\}_{t=0}^{T}$ satisfies the backward induction (72).

With the convention[21] that $\Big(\bigodot_{r=T}^{T-1} \Phi_r \Big)$ is the identity mapping, we have

[21] See Footnote 18

$\varrho_T^{\mathrm{UT}} = \langle \mathbb{G}_T \rangle$, that is, (72a). For any $\big\{A_s\big\}_t^T \in \big[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\big]_{s=t}^T$, we have:

$$
\begin{aligned}
\varrho_t^{\mathrm{N}}\Big(\big\{A_s\big\}_t^T\Big) &= \Big(\bigodot_{r=t}^{T-1}\Phi_r\Big)\Big\{\Big\langle \boxdot_{r=t}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t}^T \text{ by (104)},\\
&= \Phi_t\left\{\mathbb{G}_t\big[A_t\big], \Big(\bigodot_{r=t+1}^{T-1}\Phi_r\Big)\Big\{\Big\langle \boxdot_{r=t}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t+1}^T\right\} \text{ by (45)},\\
&= \Phi_t\left\{\mathbb{G}_t\big[A_t\big], \Big(\bigodot_{r=t+1}^{T-1}\Phi_r\Big)\Big\{\mathbb{G}_t\Big[\Big\langle \boxdot_{r=t+1}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big]\Big\}_{s=t+1}^T\right\} \text{ by (53)},\\
&= \Phi_t\left\{\mathbb{G}_t\big[A_t\big], \Big(\bigodot_{r=t+1}^{T-1}\Phi_r\Big)\mathbb{G}_t\Big[\Big\{\Big\langle \boxdot_{r=t+1}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t+1}^T\Big]\right\} \text{ by (102)},\\
&= \Phi_t\left\{\mathbb{G}_t\big[A_t\big], \mathbb{G}_t\Big[\Big(\bigodot_{r=t+1}^{T-1}\Phi_r\Big)\Big\{\Big\langle \boxdot_{r=t+1}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t+1}^T\Big]\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by commutation property (91)},\\
&= \Phi_t\left\{\mathbb{G}_t\big[A_t\big], \mathbb{G}_t\Big[\varrho_t^{\mathrm{N}}\Big(\big\{A_s\big\}_{s=t+1}^T\Big)\Big]\right\} \text{ by (104)},
\end{aligned}
$$

This ends the proof.

3.5. Applications

Now, we present applications of Theorem 17, that is, the TU case. Indeed, Theorems 9 and 12 in the nested cases NTU and NUT are less interesting because they cover cases where time-consistency is commonplace since it only depends on monotonocity assumptions. Regarding Theorem 18, it is not powerful because UT-commutation appears much more restrictive than TU-commutation: in practice, Theorem 18 only applies to linear one-step time-aggregators $\Phi\big\{c,d\big\} = \alpha c + \beta d$ (see Example 15), that obviously commute with expectations.

3.5.1. Coherent Risk Measures

We introduce a class of TU-dynamic uncertainty criteria, that are related to coherent risk measures (see Definition 6), and we show that they display time-consistency. We thus extend, to more general one-step time-aggregators, results known for the sum (see e.g. [23, 35]).

We denote by $\mathcal{P}(\mathbb{W}_t)$ the set of probabilities over $(\mathbb{W}_t, \mathcal{W}_t)$. Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{W}_0)$, $\ldots$, $\mathcal{P}_T \subset \mathcal{P}(\mathbb{W}_T)$. If A and B are sets of probabilities, then $A \otimes B$ is defined as

$$
A \otimes B = \{\mathbb{P}_A \otimes \mathbb{P}_B | \mathbb{P}_A \in A\ ,\ \ \mathbb{P}_B \in B\}\ . \tag{106}
$$

Let $(\alpha_t)_{t\in[\![0,T-1]\!]}$ and $(\beta_t)_{t\in[\![0,T-1]\!]}$ be sequences of functions, each mapping $\bar{\mathbb{R}}$ into $\mathbb{R}$, with the additional property that $\beta_t \geq 0$, for all $t \in [\![0, T-1]\!]$. We set, for all $t \in [\![0,T]\!]$,

$$\varrho^{\mathrm{co}}_{t,T}\big(\big\{A_s\big\}_{s=t}^{T}\big) = \sup_{\mathbb{P}_t\in\mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\Big[\cdots \sup_{\mathbb{P}_T\in\mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\Big[\sum_{s=t}^{T}\Big(\alpha_s\big(A_s\big)\prod_{r=t}^{s-1}\beta_r\big(A_r\big)\Big)\Big]\cdots\Big] , \tag{107}$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$, with the convention that $\alpha_T(c_T) = c_T$.

Proposition 19. *Time-consistency holds true for*

- *the dynamic uncertainty criterion* $\{\varrho^{co}_{t,T}\}_{t=0}^{T}$ *given by* (107)*,*
- *the Markov optimization problem*

$$\min_{\pi\in\Pi^{\mathrm{ad}}} \varrho^{co}_{t,T}\big(\big\{J^{x,\pi}_{t,s}\big\}_{s=t}^{T}\big) \ , \ \ \forall t\in[\![0,T]\!] \ , \ \ \forall x\in\mathbb{X}_t \ , \tag{108}$$

where $J^{x,\pi}_{t,s}(w)$ *is defined by* (27)*, as soon as there exists an admissible policy* $\pi^\sharp\in\Pi^{\mathrm{ad}}$ *such that, for all* $t\in[\![0,T-1]\!]$*, for all* $x\in\mathbb{X}_t$*,*

$$\pi^\sharp_t(x)\in\arg\min_{u\in U_t(x)}\sup_{\mathbb{P}_t\in\mathcal{P}_t}\Big\{\mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big)+\beta_t\big(J_t(x,u,\cdot)\big)V_{t+1}\circ f_t(x,u,\cdot)\Big]\Big\} ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T\in\mathcal{P}_T}\mathbb{E}_{\mathbb{P}_T}\big[J_T(x,\cdot)\big] , \tag{109a}$$

$$V_t(x) = \min_{u\in U_t(x)}\sup_{\mathbb{P}_t\in\mathcal{P}_t}\Big\{\mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + \beta_t\big(J_t(x,u,\cdot)\big)V_{t+1}\circ f_t(x,u,\cdot)\Big]\Big\} . \tag{109b}$$

Proof. The setting is that of Theorem 17 and Proposition 8, where

- the one-step time-aggregators are defined by

$$\Phi_t\big\{c_t,c_{t+1}\big\} = \alpha_t(c_t)+\beta_t(c_t)c_{t+1} \ , \qquad \forall t\in[\![0,T-1]\!] \ , \ \forall\big(c_t,c_{t+1}\big)\in\bar{\mathbb{R}}^2 \ , \tag{110a}$$

- the one-step uncertainty-aggregators are defined by

$$\mathbb{G}_t\big[C_t\big] = \sup_{\mathbb{P}_t\in\mathcal{P}_t}\mathbb{E}_{\mathbb{P}_t}\big[C_t\big] \ , \ \ \forall t\in[\![0,T-1]\!] \ , \ \ \forall C_t\in\mathcal{F}(\mathbb{W}_t;\bar{\mathbb{R}}) \ . \tag{110b}$$

The DPE (109) is the DPE (61), which holds true as soon as the assumptions of Theorem 17 hold true.

First, we prove that, for any $0\le t<s\le T$, $\mathbb{G}_s$ TU-commutes with Φ_t. Indeed, letting c_t be an extended real number in $\bar{\mathbb{R}}$ and C_s a function in $\mathcal{F}(\mathbb{W}_s;\bar{\mathbb{R}})$,

we have[22]

$$\begin{aligned}
\mathbb{G}_s\big[\Phi_t\{c_t, C_s\}\big] &= \sup_{\mathbb{P}_s \in \mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}\big[\alpha(c_t) + \beta(c_t) C_s\big]\Big\} && \text{by (110b) and (110a),} \\
&= \alpha_t(c_t) + \beta_t(c_t) \sup_{\mathbb{P}_s \in \mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}[C_s]\Big\} && \text{as } \beta_t \geq 0 \ , \\
&= \alpha_t(c_t) + \beta_t(c_t) \mathbb{G}_s[C_s] && \text{by (110b),} \\
&= \Phi_t\{c_t, \mathbb{G}_s[C_s]\} && \text{by (110a).}
\end{aligned}$$

Second, we observe that $\mathbb{G}_t$ is non-decreasing (see Definition 17), and that $c_{t+1} \in \bar{\mathbb{R}} \mapsto \Phi_t\big\{c_t, c_{t+1}\big\} = \alpha_t(c_t) + \beta_t(c_t) c_{t+1}$ is non-decreasing, for any $c_t \in \bar{\mathbb{R}}$.

This ends the proof.

The one-step uncertainty-aggregators $\mathbb{G}_t$ in (110b) correspond to a coherent risk measure, by Definition 6 and the comments that follow it.

Our result differs from [23, Theorem 2] in two ways. On the one hand, in [23], arguments are given to show that there exists an optimal Markovian policy among the set of adapted policies (that is, having a policy taking as argument the whole past uncertainties would not give a better cost than a policy taking as argument the current value of the state). We do not tackle this issue since we directly deal with policies as functions of the state. Where we suppose that there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (62) holds true, [23] gives conditions ensuring this property. On the other hand, where [23] restricts to the sum to aggregate instantaneous costs, we consider more general one-step time-aggregators Φ_t. For instance, our results applies to the product of costs.

3.5.2. Convex Risk Measures

We introduce a class of TU-dynamic uncertainty criteria, that are related to convex risk measures (see Definition 6), and we show that they display time-consistency. We consider the same setting as for coherent risk measures, with the restriction that $\beta_t \equiv 1$ and an additional data $(\Upsilon_t)_{t \in [\![0,T]\!]}$.

Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{W}_0)$, ..., $\mathcal{P}_T \subset \mathcal{P}(\mathbb{W}_T)$, and $(\Upsilon_t)_{t \in [\![0,T]\!]}$ be sequence of functions, each mapping $\mathcal{P}(\mathbb{W}_t)$ into $\bar{\mathbb{R}}$. Let $(\alpha_t)_{t \in [\![0,T]\!]}$ be sequence of functions, each mapping $\bar{\mathbb{R}}$ into $\mathbb{R}$. We set, for all $t \in [\![0,T]\!]$,

$$\varrho^{\mathrm{cx}}_{t,T}\big(\big\{A_s\big\}_t^T\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\Bigg[\cdots \sup_{\mathbb{P}_T \in \mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\Bigg[\sum_{s=t}^{T} \Big(\alpha_s(A_s) - \Upsilon_s(\mathbb{P}_s)\Big)\Bigg] \cdots\Bigg] , \quad (111)$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$, with the convention that $\alpha_T(c_T) = c_T$.

Proposition 20. *Time-consistency holds true for*

[22]This result can also be obtained by use of Proposition 24 with $I = \mathcal{P}_s$.

- *the dynamic uncertainty criterion* $\{\varrho^{cx}_{t,T}\}_{t=0}^{T}$ *given by* (111),
- *the Markov optimization problem*

$$\min_{\pi\in\Pi^{\mathrm{ad}}} \varrho^{cx}_{t,T}\big(\{J^{x,\pi}_{t,s}\}_{s=t}^{T}\big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ , \tag{112}$$

where $J^{x,\pi}_{t,s}(w)$ *is defined by* (27), *as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0, T-1]\!]$, *for all* $x \in \mathbb{X}_t$,

$$\pi^\sharp_t(x) \in \mathop{\arg\min}_{u\in U_t(x)} \sup_{\mathbb{P}_t\in\mathcal{P}_t} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + V_{t+1}\circ f_t(x,u,\cdot)\Big] - \Upsilon_t(\mathbb{P}_t)\Big\} \ ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T\in\mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\big[J_T(x,\cdot)\big] - \Upsilon_T(\mathbb{P}_T) \ , \tag{113a}$$

$$V_t(x) = \min_{u\in U_t(x)} \sup_{\mathbb{P}_t\in\mathcal{P}_t} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + V_{t+1}\circ f_t(x,u,\cdot)\Big] - \Upsilon_t(\mathbb{P}_t)\Big\} \ . \tag{113b}$$

Proof. The setting is that of Theorem 17 and Proposition 8, where

- the one-step time-aggregators are defined by

$$\Phi_t\big\{c_t, c_{t+1}\big\} = \alpha_t(c_t) + c_{t+1} \ , \ \ \forall t \in [\![0,T-1]\!] \ , \ \ \forall \big(c_t, c_{t+1}\big) \in \bar{\mathbb{R}}^2 \ , \tag{114a}$$

- the one-step uncertainty-aggregators are defined by

$$\mathbb{G}_t\big[C_t\big] = \sup_{\mathbb{P}_t\in\mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\big[C_t\big] - \Upsilon_t(\mathbb{P}_t) \ , \ \ \forall t \in [\![0,T-1]\!] \ , \ \ \forall C_t \in \mathcal{F}(\mathbb{W}_t;\bar{\mathbb{R}}) \ . \tag{114b}$$

The DPE (113) is the DPE (61), which holds true as soon as the assumptions of Theorem 17 hold true.

First, we prove that, for any $t \in [\![0, T-1]\!]$ and $s \in [\![t+1, T]\!]$, $\mathbb{G}_s$ TU-commutes with Φ_t. Indeed, letting c_t be an extended real number in $\bar{\mathbb{R}}$ and C_s a function in $\mathcal{F}(\mathbb{W}_s;\bar{\mathbb{R}})$, we have[23]

$$\begin{aligned}
\mathbb{G}_s\big[\Phi_t\{c_t, C_s\}\big] &= \sup_{\mathbb{P}_s\in\mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}\big[\alpha(c_t) + C_s\big] - \Upsilon_s(\mathbb{P}_s)\Big\} \text{ by (114b) and (114a)} \\
&= \alpha_t(c_t) + \sup_{\mathbb{P}_s\in\mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}[C_s] - \Upsilon_s(\mathbb{P}_s)\Big\} \\
&= \alpha_t(c_t) + \mathbb{G}_s[C_s] \text{ by (114b)} \\
&= \Phi_t\{c_t, \mathbb{G}_s[C_s]\} \text{ by (114a).}
\end{aligned}$$

Second, we observe that $\mathbb{G}_t$ is non-decreasing (see Definition 17), and that $c_{t+1} \in \bar{\mathbb{R}} \mapsto \Phi_t\big\{c_t, c_{t+1}\big\} = \alpha_t(c_t) + c_{t+1}$ is non-decreasing, for any $c_t \in \bar{\mathbb{R}}$.

This ends the proof.

[23]This result can also be obtained by use of Proposition 24 with $I = \mathcal{P}_s$.

The one-step uncertainty-aggregators $\mathbb{G}_t$ in (114b) correspond to a convex risk measure, by Definition 6 and the comments that follow it.

3.5.3. Worst-Case Risk Measures (Fear Operator)

A special case of coherent risk measures consists of the worst case scenario operators, also called "fear operators" and introduced in §1. For this subclass of coherent risk measures, we show that time-consistency holds for a larger class of time-aggregators than the ones above.

For any $t \in [\![0, T-1]\!]$, let $\widetilde{\mathbb{W}}_t$ be a non empty subset of $\mathbb{W}_t$, and let $\Phi_t : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ be a function which is continuous and non-decreasing in its second variable. We set, for all $t \in [\![0, T]\!]$,

$$\begin{aligned} \varrho^{\mathrm{wc}}_{t,T}\big(\big\{A_s\big\}_t^T\big) = \sup_{\{w_s\}_t^T \in \widetilde{\mathbb{W}}_t \times \cdots \times \widetilde{\mathbb{W}}_T} & \Phi_t\Bigg\{ A_t\big(\big\{w_s\big\}_t^T\big), \Phi_{t+1}\Bigg\{ \cdots , \\ & \Phi_{T-1}\Big\{ A_{T-1}(w_{T-1}, w_T), A_T(w_T)\Big\}\Bigg\}\Bigg\} , \end{aligned} \tag{115}$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$.

Proposition 21. *Time-consistency holds true for*

- *the dynamic uncertainty criterion* $\{\varrho^{wc}_{t,T}\}_{t=0}^T$ *given by* (115),
- *the Markov optimization problem*

$$\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho^{wc}_{t,T}\big(\big\{J^{x,\pi}_{t,s}\big\}_{s=t}^T\big) , \tag{116}$$

where $J^{x,\pi}_{t,s}(w)$ *is defined by* (27), *as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0, T-1]\!]$, *for all* $x \in \mathbb{X}_t$,

$$\pi^\sharp_t(x) \in \operatorname*{arg\,min}_{u \in U_t(x)} \sup_{w_t \in \widetilde{\mathbb{W}}_t} \Phi_t\Big\{ J_t\big(x, u, w_t\big), V_{t+1} \circ f_t\big(x, u, w_t\big)\Big\} ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{w_T \in \widetilde{\mathbb{W}}_T} J_T(x, w_T) , \tag{117a}$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{w_t \in \widetilde{\mathbb{W}}_t} \Phi_t\Big\{ J_t\big(x, u, w_t\big), V_{t+1} \circ f_t\big(x, u, w_t\big)\Big\} . \tag{117b}$$

Proof. The setting is that of Theorem 17 and Proposition 8, where the one-step uncertainty-aggregators are defined by

$$\mathbb{G}_t\big[C_t\big] = \sup_{w_t \in \widetilde{\mathbb{W}}_t} C_t(w_t) , \ \ \forall t \in [\![0, T-1]\!] , \ \ \forall C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}}) . \tag{118}$$

The DPE (117) is the DPE (61), which holds true as soon as the assumptions of Theorem 17 hold true.

First, we prove that, for any $t \in [\![0, T-1]\!]$ and $s \in [\![t+1, T]\!]$, $\mathbb{G}_s$ TU-commutes with Φ_t. Indeed, letting c_t be an extended real number in $\bar{\mathbb{R}}$ and C_s a function in $\mathcal{F}(\mathbb{W}_s; \bar{\mathbb{R}})$, we have[24]

$$
\begin{aligned}
\mathbb{G}_s\big[\Phi_t\{c_t, C_s\}\big] &= \sup_{w_s \in \widetilde{\mathbb{W}}_s} \Big[\Phi_t\big\{c_t, C_s(w_s)\big\}\Big] && \text{by (118),} \\
&= \Phi_t\Big\{c_t, \sup_{w \in \widetilde{\mathbb{W}}_s} \big[C_s(w_s)\big]\Big\} && \text{by continuity of } \Phi_t\{c_t, \cdot\}\ , \\
&= \Phi_t\big\{c_t, \mathbb{G}_s[C_s]\big\} && \text{by (118).}
\end{aligned}
$$

Second, we observe that $\mathbb{G}_t$ is non-decreasing (see Definition 17), and that $c_{t+1} \mapsto \Phi_t(c_t, c_{t+1})$ is non-decreasing for any $c_t \in \bar{\mathbb{R}}$, by assumption.

This ends the proof.

Note that $\varrho^{\mathrm{wc}}_{t,T}$ is simply the fear operator on the Cartesian product $\widetilde{\mathbb{W}}_t \times \cdots \times \widetilde{\mathbb{W}}_T$. An example of monotonous one-step time-aggregator is $\Phi_t\big\{c_t, c_{t+1}\big\} = \max\big\{c_t, c_{t+1}\big\}$, used in the so-called Rawls or maximin criterion [21].

3.6. Complements on TU-Commuting Aggregators

Here, we present how we can construct new TU-commuting aggregators from known TU-commuting aggregators. We do not consider UT-commutation, since we have seen that it appears much more restrictive than TU-commutation (see Example 15).

For this purpose, we consider a fixed non empty set I and a mapping Γ from $\bar{\mathbb{R}}^I$ to $\bar{\mathbb{R}}$.

3.6.1. Time-Aggregators

Let $(\Phi^i)_{i \in I}$ be a family of one-step time-aggregators. Thanks to the mapping $\Gamma : \bar{\mathbb{R}}^I \to \bar{\mathbb{R}}$, we define the one-step time-aggregator $\Gamma\big[(\Phi^i)_{i\in I}\big]$ by

$$
\Gamma\big[(\Phi^i)_{i\in I}\big]\big\{c, d\big\} = \Gamma\Big(\big\{\Phi^i\{c, d\}\big\}_{i\in I}\Big)\ , \tag{119}
$$

for all $c \in \bar{\mathbb{R}}$ and $d \in \bar{\mathbb{R}}$.

Proposition 22. *Let $t \in [\![0, T]\!]$ and $\mathbb{G}_t$ be a t-one-step uncertainty-aggregator. Suppose that*

- *$\mathbb{G}_t$ TU-commutes with ψ^i, for all $i \in I$,*
- *for all $i \in I$ and for all $C^i_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$,*

$$
\mathbb{G}_t\Big[\Gamma\big(\big\{C^i_t\big\}_{i\in I}\big)\Big] = \Gamma\bigg(\Big\{\mathbb{G}_t\big[C^i_t\big]\Big\}_{i\in I}\bigg)\ . \tag{120}
$$

[24] This result can also be obtained by use of Proposition 24 with $I = \widetilde{\mathbb{W}}_s$.

Then $\mathbb{G}_t$ TU-commutes with $\Gamma\big[(\Phi^i)_{i\in I}\big]$.

Proof. We set $\Phi = \Gamma\big[(\Phi^i)_{i\in I}\big]$. For $c \in \bar{\mathbb{R}}$ and $C_t \in \mathcal{F}(\mathbb{W}_t, \bar{\mathbb{R}})$, we have

$$\begin{aligned}
\mathbb{G}_t\Big[\Phi\big\{c, C_t\big\}\Big] &= \mathbb{G}_t\Big[\Gamma\Big(\big\{\Phi^i\{c, C_t\}\big\}_{i\in I}\Big)\Big] && \text{by definition of } \Phi \text{ in (119)},\\
&= \Gamma\bigg(\Big\{\mathbb{G}_t\big[\Phi^i\{c, C_t\}\big]\Big\}_{i\in I}\bigg) && \text{by (120) with } C_t^i = \Phi^i\{c, C_t\}\ ,\\
&= \Gamma\bigg(\Big\{\Phi^i\big\{c, \mathbb{G}_t[C_t]\big\}_{i\in I}\Big\}\bigg) && \text{by TU-commutation (83)},\\
&= \Phi\big\{c, \mathbb{G}_t[C_t]\big\} && \text{by definition of } \Phi \text{ in (119)}.
\end{aligned}$$

By Definition 25, this ends the proof.

3.6.2. Uncertainty-Aggregators

Let $t \in [\![0, T]\!]$ and $\{{\mathbb{G}_t}^i\}_{i\in I}$ be a family of t-one-step uncertainty-aggregators. Thanks to the mapping $\Gamma : \bar{\mathbb{R}}^I \to \bar{\mathbb{R}}$, we define the t-one-step uncertainty-aggregator $\Gamma\Big[\{{\mathbb{G}_t}^i\}_{i\in I}\Big]$ by

$$\Gamma\Big[\{{\mathbb{G}_t}^i\}_{i\in I}\Big]\big[C_t\big] = \Gamma\Big(\big\{{\mathbb{G}_t}^i\big[C_t\big]\big\}_{i\in I}\Big)\ ,\ \ \forall C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})\ . \tag{121}$$

We do not give the proof of the next Proposition 23, as it follows the same line as that of Proposition 22.

Proposition 23. *Let Φ be a one-step time-aggregator. Suppose that*

- *Φ TU-commutes with ${\mathbb{G}_t}^i$, for all $i \in I$,*
- *for all $c \in \bar{\mathbb{R}}$, for all $i \in I$ and for all $c^i \in \bar{\mathbb{R}}$,*

$$\Phi\Big(c, \Gamma\big(\big\{c^i\big\}_{i\in I}\big)\Big) = \Gamma\bigg(\Big\{\Phi\Big(c, \big\{c^i\big\}\Big)_{i\in I}\Big\}\bigg)\ . \tag{122}$$

Then Φ TU-commutes with $\Gamma\Big[\{{\mathbb{G}_t}^i\}_{i\in I}\Big]$.

As a corollary, we obtain the following practical result.

Proposition 24. *Let Φ be a one-step time-aggregator. Suppose that*

- *${\mathbb{G}_t}^i$ TU-commutes with Φ, for all $i \in I$,*
- *for all $c \in \bar{\mathbb{R}}$, $\Phi\big\{c, \cdot\big\}$ is continuous and non-decreasing.*[25]

[25]Instead of the continuity of $\Phi\big\{c, \cdot\big\}$, we can assume that, for all $C_t \in \mathcal{F}(\mathbb{W}_t, \bar{\mathbb{R}})$, $\sup_{i\in I} {\mathbb{G}_t}^i[C_t]$ is achieved (always true for I finite).

Then, the t-one-step uncertainty-aggregator $\sup_{i\in I} {\mathbb{G}_t}^i$ *TU-commutes with* Φ, *and so does* $\inf_{i\in I} {\mathbb{G}_t}^i$, *provided* $\inf_{i\in I} {\mathbb{G}_t}^i$ *never takes the value* $-\infty$.

Proof. We are going to show that (119) holds true, and then the proof is a straightforward application of Proposition 23. We set $\bar{\mathbb{G}}_t = \theta \sup_{i\in I} {\mathbb{G}_t}^i + (1-\theta) \inf_{i\in I} {\mathbb{G}_t}^i$, with $\theta \in [0,1]$ (only at the end, do we take $\theta \in \{0,1\}$). For any $(c, C_t) \in \bar{\mathbb{R}} \times \mathcal{F}(\mathbb{W}_t, \bar{\mathbb{R}})$, we have

$$
\begin{aligned}
\bar{\mathbb{G}}_t\Big[\Phi\big\{c, C_t\big\}\Big] = \quad & (\theta \sup_{i\in I} + (1-\theta)\inf_{i\in I}){\mathbb{G}_t}^i\Big[\Phi\big\{c, C_t\big\}\Big] \text{ by definition of } \bar{\mathbb{G}}_t \ , \\
= \quad & (\theta \sup_{i\in I} + (1-\theta)\inf_{i\in I})\Phi\Big\{c, {\mathbb{G}_t}^i\big[C_t\big]\Big\} \text{ by TU-commutation (83)}, \\
= \quad & \theta \sup_{i\in I} \Phi\Big\{c, {\mathbb{G}_t}^i\big[C_t\big]\Big\} + (1-\theta)\inf_{i\in I} \Phi\Big\{c, {\mathbb{G}_t}^i\big[C_t\big]\Big\} \ , \\
= \quad & \theta \Phi\Big\{c, \sup_{i\in I} {\mathbb{G}_t}^i\big[C_t\big]\Big\} + (1-\theta)\Phi\Big\{c, \inf_{i\in I} {\mathbb{G}_t}^i\big[C_t\big]\Big\} \ , \\
& \text{by continuity and monotonicity of } \Phi\big\{c, \cdot\big\} \ , \\
= \quad & \Phi\Big\{c, (\theta \sup_{i\in I} + (1-\theta)\inf_{i\in I}){\mathbb{G}_t}^i\big[C_t\big]\Big\} \text{ when } \theta \in \{0,1\} \ .
\end{aligned}
$$

The rest of the proof is a straightforward application of Proposition 23.

The following Proposition 25 is an easy extension of Proposition 24.

Proposition 25. *Suppose that the assumptions of Proposition 24 hold true. Let* $\underline{I}_j \subset I$, $j \in \underline{J}$ *and* $\overline{I}_j \subset I$, $j \in \overline{J}$ *be finite families of non empty subsets of* I.

- *If* Φ *is affine in its second variable, that is, if*
$$\Phi\big\{c, d\big\} = \alpha(c) + \beta(c) d \ , \tag{123}$$
and if $(\{\underline{\theta}_j\}_{j\in\underline{J}}, \{\overline{\theta}_j\}_{j\in\overline{J}})$ *are non-negative scalars that sum to one, the convex combination*
$$\sum_{j\in\underline{J}} \underline{\theta}_j \inf_{i\in\underline{I}_j} {\mathbb{G}_t}^i + \sum_{j\in\overline{J}} \overline{\theta}_j \sup_{i\in\overline{I}_j} {\mathbb{G}_t}^i \tag{124}$$
of infimum or supremum of subfamilies of $\{{\mathbb{G}_t}^i\}_{i\in I}$ *TU-commutes with* Φ, *provided* $\inf_{i\in\underline{I}_j} {\mathbb{G}_t}^i$ *never takes the value* $-\infty$.
- *If* Φ *is linear in its second variable, that is, if*
$$\Phi\big\{c, d\big\} = \beta(c) d \ , \tag{125}$$
and if $(\{\underline{\theta}_j\}_{j\in\underline{J}}, \{\overline{\theta}_j\}_{j\in\overline{J}})$ *are non-negative scalars, the combination*
$$\sum_{j\in\underline{J}} \underline{\theta}_j \inf_{i\in\underline{I}_j} {\mathbb{G}_t}^i + \sum_{j\in\overline{J}} \overline{\theta}_j \sup_{i\in\overline{I}_j} {\mathbb{G}_t}^i \tag{126}$$
of infimum or supremum of subfamilies of $\{{\mathbb{G}_t}^i\}_{i\in I}$ *TU-commutes with* Φ, *provided* $\inf_{i\in\underline{I}_j} {\mathbb{G}_t}^i$ *never takes the value* $-\infty$.

4. Extension to Markov Aggregators

Here, we extend the results of §3 to the case where we allow one-step time and uncertainty aggregators of depend on the state. The difficulty of this extension is mainly one of notations. We do not give the proofs because they follow the sketch of those in §3.2 and in §3.4. We will reap the benefits of this extension in §4.6, where we present applications.

4.1. Markov Time-Aggregators and their Composition

We allow one-step time-aggregators to depend on the state as follows (Definition 29 differs from Definition 13 only through the indexation by the state).

Definition 29. Let $t \in [\![0, T]\!]$. A *one-step Markov time-aggregator* is a family $\big\{\Phi_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}$ of one-step time-aggregators $\Phi_t^{x_t} : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ indexed by the state $x_t \in \mathbb{X}_t$.

Now, we introduce the composition of one-step Markov time-aggregators.

Definition 30. Let $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T-1}$ be a sequence of one-step Markov time-aggregators. Let $t \in [\![0, T-1]\!]$. Given a policy $\pi \in \Pi$ and $x_t \in \mathbb{X}_t$, we define the composition $\Big\langle \bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s \Big\rangle : \big[\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})\big]_t^T \to \mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})$ by

$$\left(\Big\langle \bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s \Big\rangle \Big\{ \{A_s\}_t^T \Big\} \right)(w) = \Big(\bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s^{X_{t,s}^{x_t,\pi}(w)} \Big) \Big\{ \big\{A_s\big(w\big)\big\}_t^T \Big\} , \tag{127}$$

for all scenario $w \in \mathbb{W}_{[0:T]}$, for any sequence $\{A_s\}_{s=t}^T \in \Big(\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})\Big)^{T-t+1}$, that is, where $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

Notice that the extension, to one-step Markov time-aggregators, of the composition involves the dynamical system (2) and a policy (whereas, in Definition 16, the composition is independent of the policy).

Remark 26. *Observe that we have defined* $\Big\langle \bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s \Big\rangle$, *defined over functions, but not* $\Big(\bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s \Big)$, *defined over extended reals. Observe also that the image by* $\Big\langle \bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s \Big\rangle$ *of any sequence* $c_{[t:T]}$ *of extended reals is not an extended real, but is a function:*

$$\left(\Big\langle \bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s \Big\rangle \big\{ c_{[t:T]} \big\} \right)(w) = \Big(\bigodot_{t \le s \le T-1}^{x_t,\pi} \Phi_s^{X_{t,s}^{x_t,\pi}(w)} \Big) \big\{ c_{[t:T]} \big\} . \tag{128}$$

4.2. Markov Uncertainty-Aggregators and their Composition

We allow one-step uncertainty-aggregators to depend on the state as follows (Definition 31 differs from Definition 17 only through the indexation by the state).

Definition 31. Let $t \in [\![0, T-1]\!]$. A *t-one-step Markov uncertainty-aggregator* is a family $\big\{\mathbb{G}_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}$ of t-one-step uncertainty-aggregators indexed by the state $x_t \in \mathbb{X}_t$.

We say that a sequence $\big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}\big\}_{t=0}^{T}$ of one-step Markov uncertainty-aggregators is a *chained sequence* if $\mathbb{G}_t^{x_t}$ is a t-one-step uncertainty-aggregator, for all $t \in [\![0, T]\!]$.

The extension, to one-step Markov uncertainty-aggregators, of the composition involves the dynamical system (2) and a policy (whereas, in Definition 20, the composition is independent of the policy). The formal definition is as follows.

Definition 32. Consider a chained sequence $\Big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$ of one-step Markov uncertainty-aggregators.

For a policy $\pi \in \Pi$, for $t \in [\![0, T]\!]$ and for a state $x_t \in \mathbb{X}_t$, we define the composition $\mathop{\boxdot}\limits_{t \le s \le T}^{x_t, \pi} \mathbb{G}_s$ as a functional mapping $\mathcal{F}\Big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\Big)$ into $\bar{\mathbb{R}}$, inductively given by

$$\mathop{\boxdot}\limits_{T \le s \le T}^{x_T, \pi} \mathbb{G}_s = \mathbb{G}_T^{x_T} \;, \tag{129a}$$

and then backward by, for any function $D_t \in \mathcal{F}\Big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\Big)$,

$$\begin{aligned} \Big(\mathop{\boxdot}\limits_{t \le s \le T}^{x_t, \pi} \mathbb{G}_s\Big)\big[D_t\big] = \mathbb{G}_t^{x_t}\Bigg[& w_t \mapsto \\ & \bigg(\mathop{\boxdot}\limits_{t+1 \le s \le T}^{f_t(x_t, \pi_t(x_t), w_t), \pi} \mathbb{G}_s\bigg)\Big[w_{[t+1:T]} \mapsto D_t\Big(w_t, w_{[t+1:T]}\Big)\Big]\Bigg] \;. \end{aligned} \tag{129b}$$

4.3. Time-Consistency for Nested Dynamic Uncertainty Criteria

Consider

- on the one hand, a sequence $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T-1}$ of one-step Markov time-aggregators,
- on the other hand, a chained sequence $\Big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$ of one-step Markov uncertainty-aggregators.

With these ingredients, we present two ways to design a Markov dynamic uncertainty criterion as introduced in Definition 4.

4.3.1. NTU Dynamic Markov Uncertainty Criterion

Definition 33. Let a policy $\pi \in \Pi$ be given. We construct inductively a *NTU-Markov dynamic uncertainty criterion* $\Big\{ \big\{ \varrho_{t,T}^{x_t,\pi,\mathrm{NTU}} \big\}_{x_t \in \mathbb{X}_t} \Big\}_{t=0}^{T}$ by

$$\varrho_{T}^{x_T,\pi,\mathrm{NTU}}\big(A_T\big) = \langle \mathbb{G}_T^{x_T} \rangle \big[A_T \big] \;, \tag{130a}$$

$$\varrho_{t,T}^{x_t,\pi,\mathrm{NTU}}\Big(\big\{ A_s \big\}_{s=t}^{T} \Big) = \langle \mathbb{G}_t^{x_t} \rangle \left[\Phi_t^{x_t} \left\{ A_t, \varrho_{t+1,T}^{f_t(x_t,\pi_t(x_t),\cdot),\pi,\mathrm{NTU}}\Big(\big\{ A_s \big\}_{s=t+1}^{T} \Big) \right\} \right] \;, \quad \forall t \in [\![0, T-1]\!] \;, \tag{130b}$$

for any sequence $\{x_s\}_0^T$ of states, where $x_s \in \mathbb{X}_s$.

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MNTU}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x_t,\pi,\mathrm{NTU}}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big) \;, \;\; \forall t \in [\![0, T]\!] \;, \;\; \forall x \in \mathbb{X}_t \;, \tag{131}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27).

Definition 34. We define the *value functions* inductively by the DPE

$$V_T^{\mathrm{MNTU}}(x) = \mathbb{G}_T^x \Big[J_T(x,\cdot) \Big] \;, \;\; \forall x \in \mathbb{X}_T \;, \tag{132a}$$

$$V_t^{\mathrm{MNTU}}(x) = \inf_{u \in U_t(x)} \mathbb{G}_t^x \left[\Phi_t^x \left\{ J_t(x,u,\cdot), V_{t+1}^{\mathrm{MNTU}} \circ f_t(x,u,\cdot) \right\} \right] \;, \quad \forall t \in [\![0, T-1]\!] \;, \;\; \forall x \in \mathbb{X}_t \;. \tag{132b}$$

The following Proposition 27 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{MNTU}})(x)$, for all $t \in [\![0, T]\!]$ and for all $x \in \mathbb{X}_t$, can be solved by means of the value functions in Definition 34.

Proposition 27. *Assume that*

- *for all $t \in [\![0, T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0, T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Assume that there exists[26] *an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that*

$$\pi_t^\sharp(x) \in \operatorname*{arg\,min}_{u \in U_t(x)} \mathbb{G}_t^x \left[\Phi_t^x \left\{ J_t(x,u,\cdot), V_{t+1}^{MNTU} \circ f_t(x,u,\cdot) \right\} \right] \;, \quad \forall t \in [\![0, T-1]\!] \;, \;\; \forall x \in \mathbb{X}_t \;. \tag{133}$$

[26] See Footnote 10.

Then, $\pi^\sharp$ is an optimal policy for any Problem $(\mathfrak{P}_t^{MNTU})(x)$, for all $t \in [\![0,T]\!]$ and for all $x \in \mathbb{X}_t$, and

$$V_t^{MNTU}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,NTU}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ . \qquad (134)$$

The following Theorem 28 is our main result on time-consistency in the NTU Markov case.

Theorem 28. *Assume that*

- *for all $t \in [\![0,T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0,T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Then

1. *for all policy $\pi \in \Pi$, the NTU-Markov dynamic uncertainty criterion $\Big\{ \big\{ \varrho_{t,T}^{x_t,\pi,NTU} \big\}_{x_t \in \mathbb{X}_t} \Big\}_{t=0}^{T}$ defined by (130) is time-consistent;*
2. *the Markov optimization problem $\Big\{ \big\{ (\mathfrak{P}_t^{MNTU})(x) \big\}_{x \in \mathbb{X}_t} \Big\}_{t=0}^{T}$ defined in (131) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (133) holds true.*

4.3.2. NUT Dynamic Markov Uncertainty Criterion

Definition 35. Let a policy $\pi \in \Pi$ be given. We construct inductively a *NUT-Markov dynamic uncertainty criterion* $\Big\{ \big\{ \varrho_{t,T}^{x_t,\pi,\mathrm{NUT}} \big\}_{x_t \in \mathbb{X}_t} \Big\}_{t=0}^{T}$ by

$$\varrho_T^{x_T,\pi,\mathrm{NUT}}\big(A_T\big) = \langle \mathbb{G}_T^{x_T} \rangle \big[A_T \big] \ , \qquad (135a)$$

$$\begin{aligned} \varrho_{t,T}^{x_t,\pi,\mathrm{NUT}}\big(\big\{ A_s \big\}_{s=t}^{T} \big) = \Phi_t^{x_t} \Big\{ & \langle \mathbb{G}_t^{x_t} \rangle \Big[A_t \Big], \\ & \langle \mathbb{G}_t^{x_t} \rangle \Big[\varrho_{t+1,T}^{f_t(x_t,\pi_t(x_t),\cdot),\pi,\mathrm{NUT}} \Big(\big\{ A_s \big\}_{s=t+1}^{T} \Big) \Big] \Big\} \ , \\ & \forall t \in [\![0,T-1]\!] \ , \end{aligned} \qquad (135b)$$

for any sequence $\{x_s\}_{s=0}^{T}$ of states, where $x_s \in \mathbb{X}_s$.

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MNUT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x_t,\pi,\mathrm{NUT}}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ , \quad (136)$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27).

Definition 36. We define the *value functions* inductively by the DPE

$$V_T^{\mathrm{MNUT}}(x) = \mathbb{G}_T^x\Big[J_T(x,\cdot)\Big] \ , \ \ \forall x \in \mathbb{X}_T \ , \tag{137a}$$

$$V_t^{\mathrm{MNUT}}(x) = \inf_{u \in U_t(x)} \Phi_t^x \Big\{ \mathbb{G}_t^x\big[J_t(x,u,\cdot)\big], \mathbb{G}_t^x\big[V_{t+1}^{\mathrm{MNUT}} \circ f_t(x,u,\cdot)\big] \Big\} \ , \quad \forall t \in [\![0, T-1]\!] \ , \ \ \forall x \in \mathbb{X}_t \ . \tag{137b}$$

The following Proposition 29 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{MNUT}})(x)$, for all $t \in [\![0,T]\!]$ and for all $x \in \mathbb{X}_t$, can be solved by means of the value functions in Definition 36.

Proposition 29. *Assume that*

- *for all $t \in [\![0,T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0,T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Assume that there exists[27] *an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that*

$$\pi_t^\sharp(x) \in \mathop{\arg\min}_{u \in U_t(x)} \Phi_t^x \Big\{ \mathbb{G}_t^x\big[J_t(x,u,\cdot)\big], \mathbb{G}_t^x\big[V_{t+1}^{MNUT} \circ f_t(x,u,\cdot)\big] \Big\} \ , \quad \forall t \in [\![0, T-1]\!] \ , \ \ \forall x \in \mathbb{X}_t \ . \tag{138}$$

Then, $\pi^\sharp$ is an optimal policy for any Problem $(\mathfrak{P}_t^{MNUT})(x)$, for all $t \in [\![0,T]\!]$ and for all $x \in \mathbb{X}_t$, and

$$V_t^{MNUT}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,NUT}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ . \tag{139}$$

The following Theorem 30 is our main result on time-consistency in the NUT Markov case.

Theorem 30. *Assume that*

- *for all $t \in [\![0,T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0,T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Then

1. *for all policy $\pi \in \Pi$, the NUT-Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,NUT}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$ defined by* (135) *is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}_t^{MNUT})(x)\big\}_{x \in \mathbb{X}_t}\big\}_{t=0}^{T}$ defined in* (136) *is time-consistent, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that* (138) *holds true.*

[27] See Footnote 10.

4.4. Commutation of Markov Aggregators

We extend the results on commutation obtained in §3.3 to Markov time and uncertainty aggregators. We do not give the proofs.

Consider a sequence $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T-1}$ of one-step Markov time-aggregators and a sequence $\Big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ of one-step Markov uncertainty-aggregators.

4.4.1. TU-Commutation of Markov Aggregators

The following Proposition 31 extends Proposition 14 to one-step Markov aggregators.

Proposition 31. *Suppose that, for any $0 \leq t < s \leq T$, for any states $x_t \in \mathbb{X}_t$ and $x_s \in \mathbb{X}_s$, $\mathbb{G}_s^{x_s}$ TU-commutes with $\Phi_t^{x_t}$.*

Then, for any policy $\pi \in \Pi$, any $0 \leq r < t \leq T$, any states $x_t \in \mathbb{X}_t$ and $x_r \in \mathbb{X}_r$, $\Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle$ and $\Big\langle \Phi_r^{x_r} \Big\rangle$ TU-commute, that is,

$$\Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle \Big[\Big\langle \Phi_r^{x_r} \Big\rangle \big\{ c, A \big\} \Big] = \Big\langle \Phi_r^{x_r} \Big\rangle \Big\{ c, \Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle \big[A \big] \Big\} , \tag{140}$$

for any extended scalar $c \in \bar{\mathbb{R}}$ and any function $A \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

4.4.2. UT-Commutation of Markov Aggregators

The following Proposition 32 extends Proposition 16 to one-step Markov aggregators.

Proposition 32. *Suppose that, for any $0 \leq t < s \leq T$, for any states $x_t \in \mathbb{X}_t$ and $x_s \in \mathbb{X}_s$, $\Phi_s^{x_s}$ TU-commutes with $\mathbb{G}_t^{x_t}$.*

Then, for any policy $\pi \in \Pi$, for any $0 \leq r < t \leq T$, any states $x_r \in \mathbb{X}_r$ and $x_t \in \mathbb{X}_t$, $\Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle$ TU-commutes with $\langle \mathbb{G}_r^{x_r} \rangle$, that is,

$$\Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle \Big\{ \Big\{ \langle \mathbb{G}_r^{x_r} \rangle \big[A_s \big] \Big\}_t^T \Big\} = \langle \mathbb{G}_r^{x_r} \rangle \Big[\Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle \Big\{ \big\{ A_s \big\}_t^T \Big\} \Big] , \tag{141}$$

for any $\big\{ A_s \big\}_{s=t}^T$, where $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

4.5. Time-Consistency for Non Nested Dynamic Uncertainty Criteria

4.5.1. TU Dynamic Markov Uncertainty Criterion

Definition 37. Let a policy $\pi \in \Pi$ be given. We define the *TU-Markov dynamic uncertainty criterion* $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,\mathrm{TU}}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ by[28]

$$\varrho_{t,T}^{x_t,\pi,\mathrm{TU}} = \Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle \circ \Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle \ , \ \ \forall t \in [\![0, T]\!] \ , \ \ \forall x_t \in \mathbb{X}_t \ . \tag{142}$$

[28]See Footnote 18

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MTU}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{MTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x \in \mathbb{X}_t \ , \tag{143}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27).

The following Theorem 33 is our main result on time-consistency in the TU Markov case.

Theorem 33. *Assume that*

- *for any $0 \leq s < t \leq T$, for any states $x_t \in \mathbb{X}_t$ and $x_s \in \mathbb{X}_s$, $\mathbb{G}_t^{x_t}$ TU-commutes with $\Phi_s^{x_s}$,*
- *for all $t \in [\![0, T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0, T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Then

1. *the TU-Markov dynamic uncertainty criterion $\big\{\varrho_{t,T}^{x_t,\pi,TU}\big\}_{t=0}^{T}$ defined by* (142) *is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}_t^{x_t,\pi,MTU})(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in* (143) *is time-consistent, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that* (133) *holds true, where the value functions are the $\big\{V_t^{NTU}\big\}_{t=0}^{T}$ in Definition 34.*

4.5.2. UT Dynamic Markov Uncertainty Criterion

For UT-Markov dynamic uncertainty criteria, we have to restrict the definition to the case where the sequence $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T-1}$ of one-step Markov time-aggregators is a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators (see Remark 26).

Definition 38. Let a policy $\pi \in \Pi$ be given. We define the *UT-Markov dynamic uncertainty criterion* $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,\mathrm{UT}}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ by[29]

$$\varrho_{t,T}^{x_t,\pi,\mathrm{UT}} = \Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \circ \Big\langle \mathop{\boxdot}_{t\leq s\leq T}^{x_t,\pi} \mathbb{G}_s \Big\rangle \ , \ \ \forall t \in [\![0,T]\!] \ , \ \ \forall x_t \in \mathbb{X}_t \ . \tag{144}$$

[29]See Footnote 18

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MUT}})(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{MUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) \ , \ \ \forall t\in[\![0,T]\!] \ , \ \ \forall x\in\mathbb{X}_t \ , \tag{145}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (27).

The following Theorem 34 is our main result on time-consistency in the UT Markov case.

Theorem 34. *Assume that*

- *for any $0\le s<t\le T$, for any states $x_t\in\mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ UT-commutes with Φ_s,*
- *for all $t\in[\![0,T-1]\!]$, Φ_t is non-decreasing,*
- *for all $t\in[\![0,T]\!]$, for all $x_t\in\mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Then

1. *the UT-Markov dynamic uncertainty criterion $\big\{\varrho_{t,T}^{x_t,\pi,UT}\big\}_{t=0}^{T}$ defined by (144) is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}_t^{x_t,\pi,MUT})(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in (145) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp\in\Pi^{\mathrm{ad}}$ such that (138) holds true, where the value functions are the $\big\{V_t^{NUT}\big\}_{t=0}^{T}$ in Definition 36. (where Φ_t does not depend on x_t).*

4.6. Applications

Now, we present applications of Theorem 33, that is, the TU Markov case (see the discussion introducing §3.5).

4.6.1. Coherent Markov Risk Measures

We introduce a class of TU Markov dynamic uncertainty criteria, that are related to coherent risk measures (see Definition 6), and we show that they display time-consistency.

For all $t\in[\![0,T]\!]$ and all $x_t\in\mathbb{X}_t$, let be given $\mathcal{P}_t(x_t)\subset\mathcal{P}(\mathbb{W}_t)$. Let $(\alpha_t)_{t\in[\![0,T-1]\!]}$ and $(\beta_t)_{t\in[\![0,T-1]\!]}$ be sequences of functions, each mapping $\mathbb{X}_t\times\bar{\mathbb{R}}$ into $\mathbb{R}$, with the additional property that $\beta_t\ge 0$, for all $t\in[\![0,T-1]\!]$. Notice that, to the difference with the setting in §3.5, α_t and β_t can be functions of the state x.

For a policy $\pi\in\Pi$, for $t\in[\![0,T]\!]$ and for a state $x_t\in\mathbb{X}_t$, we set

$$\begin{aligned}\varrho_{t,T}^{x_t,\pi,\mathrm{co}}(\big\{A_s\big\}_{s=t}^{T}) = \sup_{\mathbb{P}_t\in\mathcal{P}_t(x_t)} \mathbb{E}_{\mathbb{P}_t}\Bigg[\cdots \sup_{\mathbb{P}_T\in\mathcal{P}_T(X_{t,T}^{x_t,\pi})} \mathbb{E}_{\mathbb{P}_T}\Bigg[\\ \sum_{s=t}^{T}\Big(\alpha_s\big(X_{t,s}^{x_t,\pi},A_s\big)\prod_{r=t}^{s-1}\beta_r\big(X_{t,r}^{x_t,\pi},A_r\big)\Big)\Bigg]\cdots\Bigg] \ ,\end{aligned} \tag{146}$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$, with the convention that $\alpha_T(x_T,c_T)=c_T$.

Proposition 35. *Time-consistency holds true for*

- *the Markov dynamic uncertainty criterion* $\{\{\varrho_{t,T}^{x_t,\pi,co}\}_{x_t \in \mathbb{X}_t}\}_{t=0}^{T}$ *given by* (146),
- *the Markov optimization problem*

$$\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,co}\big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\big) \ , \quad \forall t \in [\![0,T]\!] \ , \quad \forall x \in \mathbb{X}_t \ , \tag{147}$$

where $J_{t,s}^{x,\pi}(w)$ *is defined by* (27), *as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0, T-1]\!]$, *for all* $x \in \mathbb{X}_t$,

$$\pi_t^\sharp(x) \in \operatorname*{arg\,min}_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,w_t)\big) + \beta_t\big(x, J_t(x,u,w_t)\big) V_{t+1} \circ f_t(x,u,w_t) \Big] \Big\} \ ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T \in \mathcal{P}_T(x)} \mathbb{E}_{\mathbb{P}_T}\big[J_T(x,\cdot)\big] \ , \tag{148a}$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,\cdot)\big) + \beta_t\big(x, J_t(x,u,\cdot)\big) V_{t+1} \circ f_t(x,u,\cdot) \Big] \Big\} \ . \tag{148b}$$

With the one-step Markov uncertainty-aggregator

$$\mathbb{G}_t^x\big[\,\cdot\,\big] = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \mathbb{E}_{\mathbb{P}_t}\big[\,\cdot\,\big] \ , \tag{149}$$

the expression $\Big\langle \mathbb{G}_t^{\boldsymbol{X}_{0,t-1}} \Big\rangle$ (see Definition 18) defines a coherent Markov risk measure (Definition 9). The associated function Ψ_t in (37) is given by

$$\Psi_t\big(v, x, u\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \mathbb{E}_{\mathbb{P}_t}\Big[v \circ f_t\big(x, u, \cdot\big) \Big] \ . \tag{150}$$

We see by (34) that, for any state $x \in \mathbb{X}_t$, and any control $u \in \mathbb{U}_t$, the function $v \mapsto \Psi_t\big(v,x,u\big)$, is a coherent risk measure (see Definition 9).

4.6.2. Convex Markov Risk Measures

We introduce a class of TU-dynamic uncertainty criteria, that are related to convex risk measures (see Definition 6), and we show that they display time-consistency. We consider the same setting as for coherent risk measures, with the restriction that $\beta_t \equiv 1$ and an additional data $(\Upsilon_t)_{t \in [\![0,T]\!]}$.

For all $t \in [\![0,T]\!]$ and all $x_t \in \mathbb{X}_t$, let be given $\mathcal{P}_t(x_t) \subset \mathcal{P}(\mathbb{W}_t)$. Let $(\Upsilon_t)_{t \in [\![0,T]\!]}$ be a sequence of functions Υ_t mapping $\mathbb{X}_t \times \mathcal{P}(\mathbb{W}_t)$ into $\bar{\mathbb{R}}$. Let $(\alpha_t)_{t \in [\![0,T]\!]}$ be a sequence of functions α_t mapping $\mathbb{X}_t \times \bar{\mathbb{R}}$ into $\mathbb{R}$. Notice that, to the difference with the setting in §3.5, α_t and Υ_t can be functions of the state x.

For a policy $\pi \in \Pi$, a time $t \in [\![0, T]\!]$ and a state $x_t \in \mathbb{X}_t$, we set

$$\begin{aligned} \varrho_{t,T}^{x_t,\pi,\mathrm{cx}}(\{A_s\}_{s=t}^T) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x_t)} \mathbb{E}_{\mathbb{P}_t}\Bigg[\cdots \sup_{\mathbb{P}_T \in \mathcal{P}_T(x_T)} \mathbb{E}_{\mathbb{P}_T}\Bigg[\\ \sum_{s=t}^{T} \Big(\alpha_s\big(x_s, A_s\big) - \Upsilon_s(x_s, \mathbb{P}_s) \Big) \Bigg] \cdots \Bigg] , \end{aligned} \tag{151}$$

for any adapted uncertain process $\{A_t\}_0^T$, with the convention that $\alpha_T(c_T) = c_T$.

Proposition 36. *Time-consistency holds true for*

- *the dynamic uncertainty criterion* $\{\{\varrho_{t,T}^{x_t,\pi,cx}\}_{x_t \in \mathbb{X}_t}\}_{t=0}^T$ *given by* (151),
- *the Markov optimization problem*

$$\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,cx}(\{J_{t,s}^{x,\pi}\}_{s=t}^T) \ , \quad \forall t \in [\![0, T]\!] \ , \quad \forall x \in \mathbb{X}_t \ , \tag{152}$$

where $J_{t,s}^{x,\pi}(w)$ *is defined by* (27), *as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0, T-1]\!]$, *for all* $x \in \mathbb{X}_t$,

$$\pi_t^\sharp(x) \in \arg\min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,\cdot)\big) + V_{t+1} \circ f_t(x,u,\cdot) \Big] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \ ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T \in \mathcal{P}_T(x)} \Big\{ \mathbb{E}_{\mathbb{P}_T}\Big[\alpha_T\big(x, J_T(x,\cdot)\big) \Big] - \Upsilon_T(x, \mathbb{P}_T) \Big\} \ , \tag{153a}$$

$$\begin{aligned} V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,\cdot)\big) \\ + V_{t+1} \circ f_t(x,u,\cdot) \Big] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \ . \end{aligned} \tag{153b}$$

With the one-step Markov uncertainty-aggregator

$$\mathbb{G}_t^x[\,\cdot\,] = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}[\,\cdot\,] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \ , \tag{154}$$

the expression $\Big\langle \mathbb{G}_t^{\boldsymbol{X}_{0,t-1}} \Big\rangle$ (see Definition 18) defines a convex Markov risk measure (Definition 9). The associated function Ψ_t in (37) is given by

$$\Psi_t\big(v, x, u\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[v \circ f_t\big(x, u, \boldsymbol{W}_t\big) \Big] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \ . \tag{155}$$

We see by (34) that, for any state $x \in \mathbb{X}_t$, and any control $u \in \mathbb{U}_t$, the function $v \mapsto \Psi_t\big(v, x, u\big)$, is a convex risk measure (see Definition 9).

5. Discussion

We discuss how our assumptions and results in §3 relate to other results in the literature on time-consistency for dynamic risk measures

First, we examine the connections between time-consistency for Markov dynamic uncertainty criteria and the existence of a DPE. When we analyze the literature on time-consistency for risk measures with our tools (aggregators), we observe that

- most, if not all results, are obtained for the specific case of linear one-step time-aggregators $\Phi_t\{c_t, c_{t+1}\} = c_t + c_{t+1}$,
- a key ingredient to obtain time-consistency is an equation like (156a), which corresponds to the commutation of one-step uncertainty-aggregators with the sum (that is, with the linear one-step time-aggregators actually used).

Therefore, Theorems 9, 12, 17, 18 in §3 provide an umbrella for most of the results establishing time-consistency for dynamic risk measures, and yields extensions to more general time-aggregators than the sum. In [32], time-consistency for dynamic risk measures is not defined by a monotonicity property like in [23] but in line with the existence of a DPE. In [33], the time-consistency property is comparable to Definition 11, though being restricted to the multiplicative time-aggregator.

We discuss to some extent [23] where time-consistency for dynamic risk measures plus an additional assumption like (156a) lead to the existence of a DPE, within the original framework of *Markov risk measures* sketched above. Here is the statement of Theorem 1 in [23], with the notations of §2.2.

Theorem 37 ([23]). *Suppose that a dynamic risk measure $\{\rho_{t,T}\}_{t=0}^{T}$ satisfies, for all $t \in [\![0, T]\!]$, and all $\boldsymbol{A}_t \in \mathcal{L}_t$ the conditions*

$$\rho_{t,T}\Big(\big\{\boldsymbol{A}_s\big\}_{s=t}^{T}\Big) = \boldsymbol{A}_t + \rho_{t,T}\Big(\big\{0, \boldsymbol{A}_{t+1}, \cdots, \boldsymbol{A}_T\big\}\Big) , \tag{156a}$$

$$\rho_{t,T}\Big(\big\{0\big\}_{s=t}^{T}\Big) = 0 . \tag{156b}$$

Then ρ is time-consistent iff, for all $0 \leq s \leq t \leq T$ and all $\{\boldsymbol{A}_s\}_0^T \in \mathcal{L}_{0,T}$, the following identity is true:

$$\rho_{s,T}\Big(\big\{\boldsymbol{A}_r\big\}_{r=s}^{T}\Big) = \rho_{s,t}\Big(\big\{\boldsymbol{A}_r\big\}_{r=s}^{t}, \rho_{t,T}\big(\big\{\boldsymbol{A}_r\big\}_{r=t}^{T}\big)\Big) . \tag{157}$$

In [23, Section 5], the finite horizon problem corresponds to Problem (95), starting at $t = 0$, where the one-step uncertainty aggregator $\mathbb{G}_t$ in (95) corresponds to the one-step conditional risk measure ρ_t, the one-step time-aggregator Φ_t in (95) corresponds to the sum, and the cost J_t in (95) is denoted c_t in [23]. Commutation of the one-step time-aggregators Φ_t and the one-step uncertainty-aggregators $\mathbb{G}_s$ is ensured through the equivariance translation property (156a) of a coherent measure of risk. Monotonicity of the uncertainty aggregator $\mathbb{G}_s$ corresponds to the monotonicity property of a coherent risk measure, and monotonicity of the time aggregator is obvious. Thus, Theorem 17 leads to the same DPE as [23, Theorem 2].

Let us now focus on the differences between [23] and our results. In [23], arguments are given to show that there exists an optimal Markovian policy among the set of adapted policies (that is, having a policy taking as argument the whole past uncertainties would not give a better cost than a policy taking as argument the current value of the state). We do not tackle this issue since we directly deal with policies as functions of the state. Where we suppose that there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (62) holds true, [23] gives conditions ensuring this property. Finally, where [23] restricts to the sum to aggregate instantaneous costs, we consider more general one-step time-aggregators Φ_t. Moreover where we give a sufficient condition for a Markovian policy to be optimal, [23] gives a set of assumptions such that this sufficient condition is also necessary (typically assumption ensuring that minimums are attained).

Second, we discuss the possibility to modify a Markov optimization problem or a dynamic risk measure, in order to make it time-consistent (if it were not originally). When sequences of optimization problems are not time-consistent with the original "state", they can be made time-consistent by extending the state. In [6], this is done for a sequence of optimization problem under a chance constraint. In [22, Example 1], the sum of AV@R of costs is considered (given by the dynamic risk measure defined in 1.2.2 and labeled (TU)). This formulation is not time consistent. However, exploiting the formulation (19) of AV@R, we suggest to extend the state and add the variables $\{r_s\}_0^T$ so that, after transformation, we obtain a problem with expectation as uncertainty aggregator, and sum as time aggregator, thus yielding time-consistency. In [31], it is shown how a large class of possibly time-inconsistent dynamic risk measures, called spectral risk measures and constructed as a convex combination of AV@R, can be made time-consistent by what we interpret as an extension of the state.

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